Undirected edge geography

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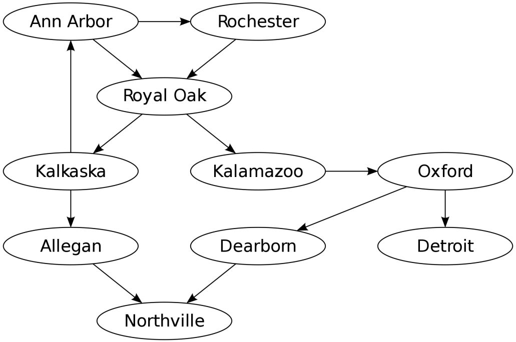
Generalized Geography

Rules

- The board is a directed graph,
- 2 players alternately move a token,
- The token can only be moved along an edge from one vertex to another,
- During every move the visited vertex is removed,
- The player who has no legal move loses.



Variations

Directed/Undirected

Whether the edges of the graph are directed or not.

Vertex/Edge

Remove visited vertices or edges after every move.

Rooted/Unrooted

The first player starts from assigned position or chooses one arbitrarily.

Variations

Restricted graph

Planer, bipartite, restricted maximum degree...

Multiple tokens

Tokens block or annihilate one another, two colors...

Edge Geography

Undirected vertex geography

Theorem

We can determine the winner of a UVG game in polynomial time

Undirected vertex geography

Matching

A matching in a graph is a set of independent edges.

Maximum matching

A matching of maximum cardinality.

Saturation

A matching M saturates a vertex v if v is incident to some edge in M.

Undirected vertex geography

Lemma

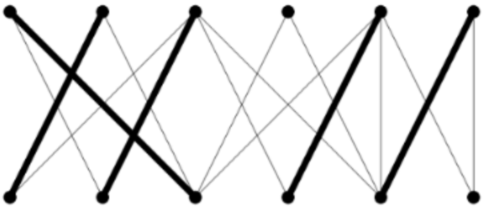
Let G be a graph and v a distinguished vertex of G. Then the game UVG starting at position (G, v) is a first player win if and only if every maximum matching of G saturates v.

Undirected vertex geography

=>

Suppose that some maximum matching M does not saturate the starting vertex v.

The winning strategy for the second player is to move along edges from M.

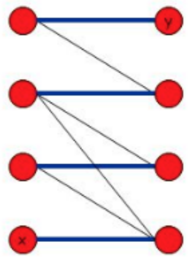


Undirected vertex geography

<=

Every maximum matching saturates v v.

The winning strategy for the first player is to choose any maximum matching M and move along edges from M. □



Undirected vertex geography

Corollary

In unrooted UVG the second player has a winning strategy if and only if the graph has a perfect matching.

Undirected vertex geography

Maximum matching can be solved in plynomial time (e.g. blossom algorithm) $=>$ undirected vertex geography can be solved in polynomial time.

Undirected edge geography

Theorem

The problem of determining if the second player wins an UEG game starting from given vertex v of G is PSPACE-complete.

Undirected edge geography

=>

The game can last no more than $|E|$ moves.

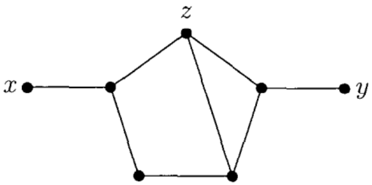
Undirected edge geography

$<=$

A polynomial time transform from Directed Edge Geography (PSPACE-complete)

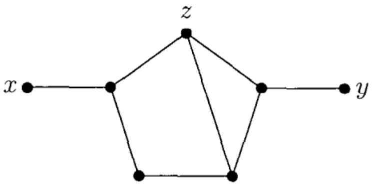
INPUT: arbitrary rooted directed graph (D, w).
We construct a rooted undirected (G, v) - instance of UEG...

Create G by replacing every directed edge $x \to y$ in D with a copy of A (pictured on the right). $v = w$



Graph A

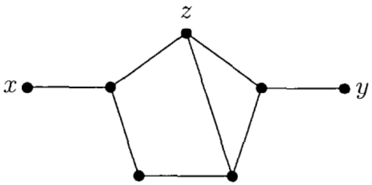
No (optimal) player ever play from y into A - the opponent can force a local victory by going to z.



Graph A

Undirected edge geography

A player can safely pass the other way around in exactly 4 moves - no player can win in A starting from x.



Graph A

Undirected edge geography

DEG is known to be PSPACE-complete => UEG is PSPACE complete.
DEG remain PSPACE even for planar graphs with maximum vertex degree of 3.

Special cases

Even kernel

Def

An even kernel for G is a nonempty set S of vertices such that

1. no two elements of S are adjacent,
2. every vertex not in S is adjacent to an even number (possibly 0) of vertices in S.

Even kernel

Theorem

IF S is an even kernel for G and $v \in S$, then the second player wins a game that starts from v.

Strategy for the second player: always move to a vertex in S.

Even kernel

- the first player always moves to a vertex not in S,
- each vertex not in S has an even number of neighbours in S.

Even kernel

Corollary

Nonadjacent vertices v and w in a graph G have the same neighbour $=>$ the second player wins a game that starts form v.

$S = \{v, w\}$ is an even kernel for G.

Even kernel

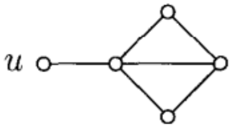
Corollary

For $n \geq 2$, the first player wins in a K_n graph.

The first player moves from v to (any) w. v and w have the same neighbours in $K_n - vw$.

Even kernel

This theorem works only one way - example:



Bipartite graphs

Bipartite adjacency matrix

For bipartite graph G with parts X and Y a bipartite adjacency matrix is an $|X|$-by-$|Y|$ matrix whose i, j entry is 1 if $x_i y_j$ is an edge of G and 0 otherwise.

- line of a matrix - a row or a column,
- lines are perpendicular if one is a row and one is a column, otherwise they are parallel.

Bipartite graphs

UEG on a matrix

A legal move consists of choosing a 1 in the distinguished line of M, changing it to a 0, and identifying the perpendicular line through this entry to be the new distinguished line.

Bipartite graphs

Theorem

Given a binary matrix M and a distinguished line l the second player wins a game that starts form l if and only if l is in the span over $GF(2)$ of the other lines parallel to it.

Bipartite graphs

=>

Suppose (without loss of generality), that l is the first row r_1 and is in the span of the remaining rows, that is

$$\sum_{i \in Q} r_i = 0$$

where Q is some subset of rows with $l \in Q$.

Set S of vertices of G associated with rows $r_i, i \in Q$ is an even kernel for G. The second player wins.

Bipartite graphs

$<=$

Suppose that r_1 is linearly independent of the remaining rows. We show that there is a 1 in row r_1, say $m_{1,j} = 1$ which when changed to a 0 makes column c_j dependent on the other columns.

Bipartite graphs

Let M^* be a matrix obtained from M by adding a column $e_1 = (1, 0, 0, ..., 0)^T$.

r_1 is independent => M^* has the same rank as M => has the same column rank as M, that is:

$$e_1 = \sum_j b_j c_j$$

Choose an index k so that $b_k = 1$ and the first entry of c_k is 1 (k must exist since the first entry of e_1 is one).

$$0 = c_k + e_1 = \sum_{j \neq k} b_j c_j$$

The column is dependent on the remaining columns.

Bipartite graphs

By the first part of this prove the opponent now faces a losing position - this is a win for the first player. □

Bipartite graphs

Corollary

Let v be a vertex of a bipartite graph G. The second player wins a game that starts from v if and only if v is in an even kernel of G.

Bipartite graphs

Corollary

There is a polynomial-time algorithm for UEG in case the rooted graph is bipartite.

A simple Gaussian elimination suffices.

Bipartite graphs

Corollary

If G is a bipartite Eulerian graph and v is any vertex, then the second player wins a game that starts from v.

All vertices in the two parts of G have even degree, so each part is an even kernel.

Bipartite graphs

Corollary

If G is a bipartite graph with parts X and Y, v is a vertex in X, and every vertex in Y has even degree, then the second player wins a game that starts from v.

Bipartite graphs

Corollary

Let G be a bipartite graph with bipartite adjacency matrix M. The second player has a winning strategy in unrooted UEG if and only if M is invertible (i.e., square and nonsingular) over GF(2).

n-cube graph

n-cube graph

The n-cube Qn is the graph whose vertex set is the set of $\{0, I\}$ sequences of length n with an edge between two such sequences if they differ in exactly one position.

n-cube graph

Theorem

Let v be any vertex in the n-cube Q_n, then the second player wins a game that starts from v if and only if n is even.

n-cube graph

=>

Every Q_n is bipartite. For even n it is also Eulerian.

n-cube graph

$<=$

Suppose n is odd. Without loss of generality, let $v = (0, 0, ..., 0)$.
Let $w = (1, 0, ..., 1)$ and let S be a set of $\{0, 1\}$ sequences of length n with exactly single 1.

S is an even kernel for $(Q_n - vw, w) =>$ the second player wins in $(Q_n - vw, w)$.

The winning strategy for the first player in Q_n is to always move into S. $\square$

$m \times n$-grid graph

$m \times n$-grid

The $m \times n$-grid $G_{m\times n}$ is the graph whose vertex set is $\{1, 2, ..., m\} \times \{1, 2, ..., n\}$ with an edge between (i, j) and (k, l) if and only if $|i - k| + |j - l| = 1$.

$m \times n$-grid graph

Theorem

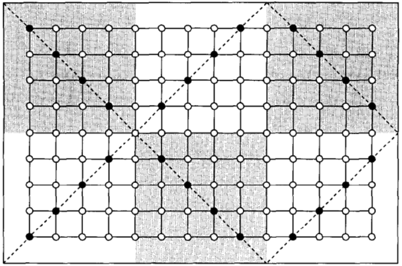
Take $m, n \geq 2$ and let $v = (1, 1)$. Then the second player wins a game on $G_{m \times n}$ that starts from v if and only if $gcd(m+1, n+1) \neq 1$.

=>

Let $d > 1$ be a common divisor of $m+1$ and $n+1$. Let

$$S = \{(i, j) : d \nmid i \wedge d \nmid j \wedge (2d \mid (i-j) \vee 2d \mid (i+j)\}$$

S is an even kernel for $G_{m \times n}$ so the second player wins a game on $G_{m \times n}$ that starts from $(1, 1)$



$m \times n$-grid graph

$<=$

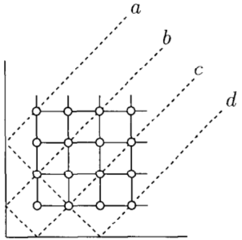
Suppose $gcd(m+1, n+1) = 1$. Either m or n is even, without loss of generality let m be even.

We show that at least one of possible opening moves leads to a second-player-wins situation.

$m \times n$-grid graph

Imagine a billiard ball rolling on an $(m+1)$-by-$(n+1)$ billiard table (with corners coordinatized by (O, O), $(m+l, O)$, $(0, n+l)$, and $(m+1, n+1)$).

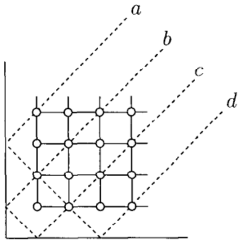
The billiard ball begins its journey at corner $(m+1, 0)$ always maintaining a 45° angle with the sides of the table. As it travels, the ball bounces off the edges of the table. After traveling through $lcm(m+1, n+1) = (m+1)(n+1)$ steps, it is "absorbed" in another corner.



$m \times n$-grid graph

- The ball's trajectory can take it through any given vertex of the grid graph at most twice,
- The ball can travel through only half of the vertices $((i,j) : i + j = 1 \ (mod\, 2))$

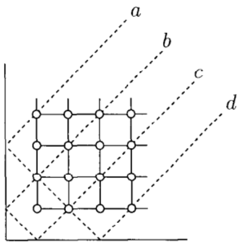
The orbit visits every vertex $(i,j) : i + j = 1 \ (mod\, 2)$ exactly twice.



$m \times n$-grid graph

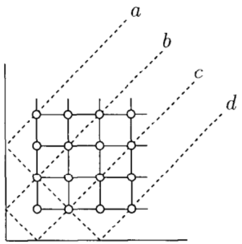
Four portions of the trajectory are labeled by a, b, c and d. Two of these four portions lead to corners of the table.

It is not the case that both the a and d portions lead to a corner (for otherwise we would only visit $(1, 2)$ and $(2, 1)$ once); likewise at most one of b and c lead to a corner.

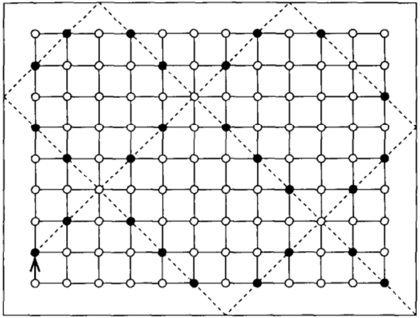


Therefore, exactly one of portions b or c lead to a corner; let us say it is portion b.

Let S be the set of all vertices which are traversed an odd number of times on portion b.



Observe that every vertex not in S (an independent set) is adjacent to an even number of members of S except vertex $(1, 1)$.
A move from $(1, 1)$ to $(1, 2)$ results in wining position for the second player $=>$ this is a winning move for the first player. $\square$



Problems

- Even Kernel Computational Complexity,
- Unrooted Undirected Edge Geography,
- Random graphs,
- Grids in general,
- Partizan Versions.

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References

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