ARRIVAL game

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Sources

- *ARRIVAL: A zero-player graph game in NP $\cap$ coNP* (2017) – by J. Dohrau, B. Gartner, M. Kohler, J. Matoušek, E. Welzl
- *ARRIVAL: Next Stop in CLS* (2018) – by B. Gartner, T. Dueholm Hansen, P. Hubáček, K. Král, H. Mosaad, V Slívová

Introduction

Suppose that a train is running along a railway network of a special nature: every time the train traverses a switch, the switch will change its position immediately afterwards. Hence, the next time the train traverses the same switch, the other direction will be taken, so that directions alternate with each traversal of the switch.

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What is the complexity of deciding so?

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- V is a set of vertices
- $s_0, s_1 : V \mapsto V$
- $E = \{(v, s_0(v) : v \in V)\} \cup \{(v, s_1(v) : v \in V)\}$, with loops (v, v) allowed

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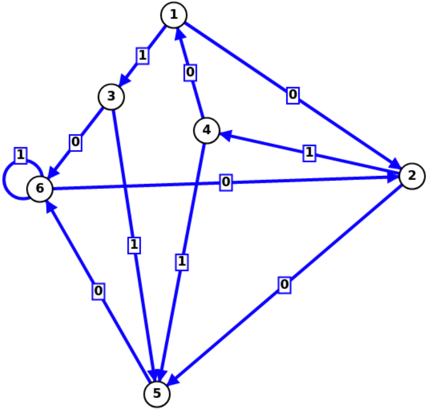
- $E^+(v)$ denotes the set of outgoing edges from v, $E^-(v)$ denotes the set of incoming edges

Example of switch graph



Running train

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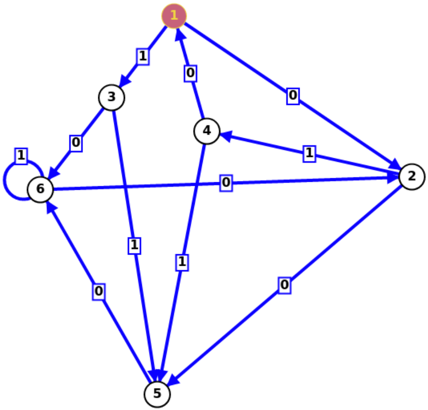
```
procedure Run(G, o, d)
    v := o
    while v ≠ d do
        w := s_curr[v]
        swap (s_curr[v], s_next[v])
        v := w
    end while
end procedure
```

where initially $\texttt{s\_curr}[v] = s_0(v), \texttt{s\_next}[v] = s_1(v)$.

Example run



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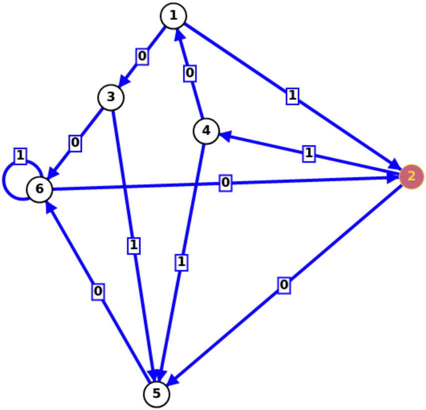

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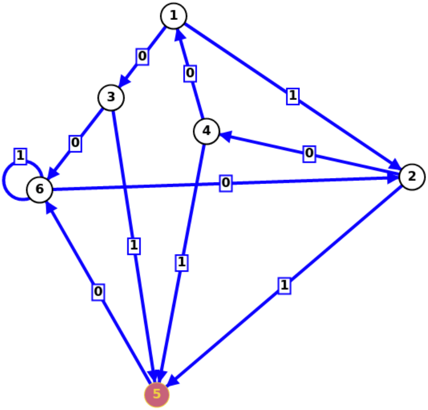

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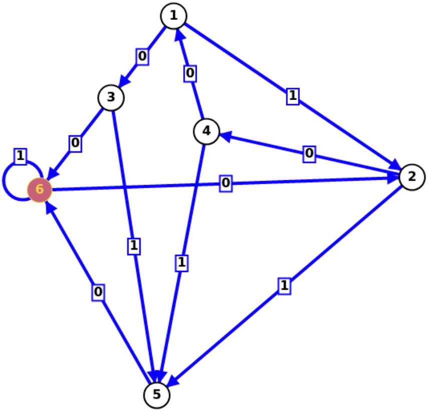

ARRIVAL

Problem *ARRIVAL* is to decide whether procedure $\texttt{Run}(G, o, d)$ terminates for a given switch graph $G = (V, E, s_0, s_1)$ and $o, d \in V$.

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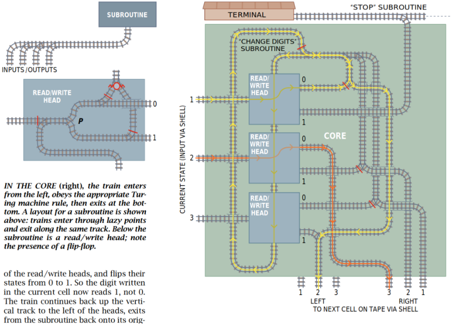

IN THE CORE (right), *the train enters from the left, obeys the appropriate Turing machine rule, then exits at the bottom. A layout for a subroutine is shown above: trains enter through lazy points and exit along the same track. Below the subroutine is a read/write head; note the presence of a flip-flop.*

of the read/write heads, and flips their states from 0 to 1. So the digit written in the current cell now reads 1, not 0. The train continues back up the vertical track to the left of the heads, exits from the subroutine back onto its orig-

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Theorem 1.

Problem ARRIVAL is decidable.

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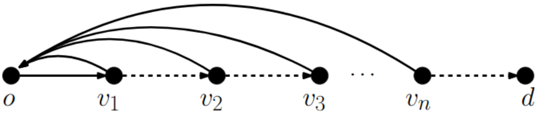

Figure: Switch graph, on which *RUN* procedure takes exponential number of steps. Solid edges point to the even successors, dashed point to the odd.

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- natural witness – *run profile*
- fake run profiles may fool the verifier

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$$\forall_{v \in V} \sum_{e \in E^+(v)} x(e) - \sum_{e \in E^-(v)} x(e) = \begin{cases} 1, & v = o \\ -1, & v = d \\ 0, & otherwise \end{cases}$$

$$\forall_{v \in V}\ 0 \leq x((v, s_1(v))) \leq x((v, s_0(v))) \leq x((v, s_1(v))) + 1$$

Run profile vs switching flow

Observation 1.

Let $G = (V, E, s_0, s_1)$ *be a switch graph, and let* $o, d \in V, o \neq d$*, such that* $\mathit{Run}(G, o, d)$ *terminates. Let* $x(G, o, d) : E \mapsto \mathbb{N}$ *(the run profile) be the function that assigns to each edge the number of times it has been traversed during* $\mathit{Run}(G, o, d)$*. Then* $x(G, o, d)$ *is a switching flow.*

Fake flows

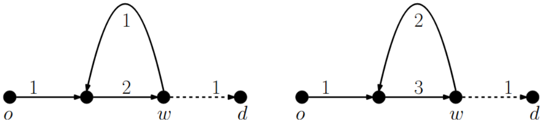

Figure 2: Run profile (left) and fake run profile (right); both are switching flows. Solid edges point to even or unique successors, dashed edges to odd successors.

Switching flow is enough

Lemma 1.

Let $G = (V, E, s_0, s_1)$ be a switch graph, and let $o, d \in V, o \neq d$. If there exists a switching flow x, then $\mathcal{Run}(G, o, d)$ terminates, and $x(G, o, d) \leq x$ (componentwise).

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- during the run, flow conservation (w.r.t. to the remaining pebbles) always holds, except at d, and at the current vertex which has one more pebble on its outgoing edges
- by alternation, starting with the even successor, the numbers of pebbles on $(v, s_0(v))$ and $(v, s_1(v))$ always differ by at most one, for every vertex v

So *NP*

Theorem 2.

Problem ARRIVAL is in NP.

coNP

Theorem 3.

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coNP

- dead vertices (dead ends)

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- dead edges

coNP

- dead vertices (dead ends)
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- hopeful edges

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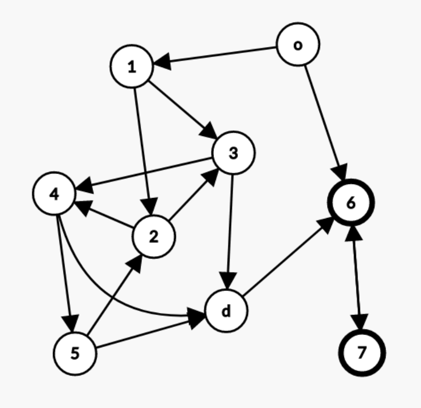

Lemmas

Lemma 2.

Let $G = (V, E, s_0, s_1)$ *be a switch graph,* $o, d \in V, o \neq d$*, and let* $e = (v, w) \in E$ *be a hopeful edge of desperation* k*. Then* $\mathtt{Run}(G, o, d)$ *will traverse* e *at most* $2^k + 1 - 1$ *times.*

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Let $G = (V, E, s_0, s_1)$ *be a switch graph, and let* $o, d \in V, o \neq d$*. If* $\mathit{Run}(G, o, d)$ *does not terminate, it will reach a dead end.*

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Proof based on *Lemma 1*.

coNP

Theorem 3.

Problem ARRIVAL is in coNP.

Given instance (G, o, d) we will construct (in polynomial time) another instance $(\bar{G}, o, \bar{d})$ such that `Run` on the first one terminates iff it does not terminate on the second one.

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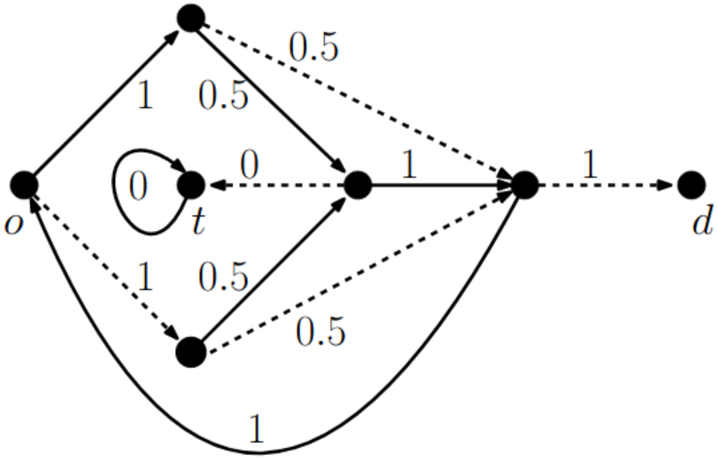
- for the rest: $\bar{s_0}(v) = s_0(v), \bar{s_1}(v) = s_1(v)$

Theorem 4.

Let $G = (V, E, s_0, s_1)$ be a switch graph, and let $o, d \in V, o \neq d$. $\mathcal{Run}(G, o, d)$ terminates if and only if there exists an integer solution satisfying the constraints (1) and (2). In this case, the run profile $x(G, o, d)$ is the unique integer solution that minimizes the linear objective function $\sum x = \sum_{e \in E} x(e)$ subject to the constraints (1) and (2).

No integer solution



Further results

- $ARRIVAL \in UP \cap coUP$
- $S - ARRIVAL \in PLS$
- $S - ARRIVAL \in CLS$