Collatz conjecture

by Mikołaj Twaróg

April 2020
Proposed by Lothar Collatz in 1937.
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Many mathematicians tried solving it.
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Many mathematicians tried solving it.
Soviet conspiracy aimed to slow down mathematical progress?
“Mathematics is not yet ready for such problems”
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Paul Erdős
Statement

**Col function**

\[ \text{Col}(n) = \begin{cases} 
3n + 1, & \text{for } 2 \nmid n \\
\frac{n}{2}, & \text{for } 2 \mid n 
\end{cases} \]

Collatz conjecture

For every \( n \in \mathbb{N} \) we eventually reach 1 by repeatedly applying \( \text{Col} \) to \( n \).
Statement

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Collatz conjecture

For every \( n \in \mathbb{N} \) we eventually reach 1 by repeatedly applying Col to \( n \).
Examples

- For \( n = 9 \) we get the sequence:
  
  \[ 9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1 \]
Examples

- For $n = 9$ we get the sequence:
  9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1

- For $n = 27$ we get a sequence consisting of 111 numbers and biggest
  number $9232$. 

Highest number of steps

less than 10 is 9, which has 19 steps,
less than 100 is 97, which has 118 steps,
less than 1000 is 871, which has 178 steps,
less than $10^4$ is 6171, which has 261 steps,
less than $10^5$ is 77031, which has 350 steps,
less than $10^6$ is 837799, which has 524 steps,
less than $10^7$ is 8400511, which has 685 steps,
less than $10^8$ is 63728127, which has 949 steps,
less than $10^9$ is 670617279, which has 986 steps,
less than $10^{10}$ is 9780657630, which has 1132 steps,
less than $10^{11}$ is 75128138247, which has 1228 steps,
less than $10^{12}$ is 989345275647, which has 1348 steps,
less than $10^{13}$ is 7887663552367, which has 1563 steps,
less than $10^{14}$ is 80867137596217, which has 1662 steps,
less than $10^{15}$ is 942488749153153, which has 1862 steps,
less than $10^{16}$ is 7579309213675935, which has 1958 steps
Number of steps for numbers from 1 to $10^8$
Trees

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Collatz conjecture

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Disproving

finding a cycle

\[ \lim_{x \to +\infty} C(x) = +\infty \]

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Collatz conjecture

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Disproving

- finding a cycle
Disproving

- finding a cycle
- finding \( n, \lim_{x \to +\infty} Col^x(n) = +\infty \)
Cycle is a sequence \((a_0, ..., a_k)\) of distinct numbers where \(Col(a_i) = a_{i+1}\) for \(0 < i < k\) and \(Col(a_k) = a_0\).
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**Eliahou Shalom (1993)**

Period \(p\) of any non-trivial cycle is of the form

\[ p = 301994a + 17087915b + 85137581c \]

where \(b \geq 1\) and \(ac = 0\).
Cycle is a sequence \((a_0, \ldots, a_k)\) of distinct numbers where \(Col(a_i) = a_{i+1}\) for \(0 < i < k\) and \(Col(a_k) = a_0\).

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Period \(p\) of any non-trivial cycle is of the form

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\]

where \(b \geq 1\) and \(ac = 0\)

Largest found lower bound for cycle length is 17026679261.
Supporting arguments

results were calculated for all $n \leq 10^2$ on average the next odd number is $\frac{3}{4}$ of a previous one

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Supporting arguments

- results were calculated for all $n \leq 10^{20}$
- on average the next odd number is $\frac{3}{4}$ of a previous one
Jeffrey C. Lagarias

"The Ultimate Challenge: The 3x + 1 Problem"
Jeffrey C. Lagarias

"The Ultimate Challenge: The 3x + 1 Problem"

"Now I know lots more about the problem, and I’d say it’s still impossible"
Different approach

Natural density

For $M \subseteq \mathbb{N}$ its density is defined as:

$$\lim_{x \to +\infty} \frac{\text{card}\{y \in M|y < x\}}{x}$$
Different approach

Riho Terras (1976)

The set $M = \{x \in \mathbb{N} | (\exists n)(Col^n(x) < x)\}$ has natural density of 1.
Different approach

Riho Terras (1976)

The set \( M = \{ x \in \mathbb{N} | (\exists n)(Col^n(x) < x) \} \) has natural density of 1.

Ivan Korec (1994)

For every \( c > \log_4 3 \) the set \( M = \{ x \in \mathbb{N} | (\exists n)(Col^n(x) < x^c) \} \) has natural density of 1.
"Almost all orbits of the Collatz map attain almost bounded values"

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Almost all orbits of the Collatz map attain almost bounded values.

Let $f : \mathbb{N} + 1 \to \mathbb{R}$ be any function with $\lim_{N \to \infty} f(N) = +\infty$. Then one has $Col_{\text{min}}(N) < f(N)$ for almost all $N \in \mathbb{N} + 1$. 

\[ Col_{\text{min}}(N) = \inf_{n \in \mathbb{N}} Col^n(N) \]
How did he do it?

Changing density from natural to logarithmic.

Finding similarities to PDEs.
How did he do it?

- Changing density from natural to logarithmic.
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- Changing density from natural to logarithmic.
- Finding similarities to PDEs.
Partial differential equation (PDE) is a differential equation that contains unknown multivariable functions and their partial derivatives.
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PDEs can be used to describe a wide variety of phenomena such as sound, heat, diffusion, electrostatics, electrodynamics, fluid dynamics, elasticity, gravitation and quantum mechanics.
Similarities

Repeating the same process on a value to understand what happens in the future.

Statistical way of studying the long-term behavior of a small number of starting values.
Repeating the same process on a value to understand what happens in the future.
Similarities

- Repeating the same process on a value to understand what happens in the future.
- Statistical way of studying the long-term behavior of a small number of starting values.
Finding the right numbers

Problems:
- numbers getting smaller
- numbers getting clumped together
Finding the right numbers

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Problems:

- numbers getting smaller
- numbers getting clumped together
“You could get obsessed with these big famous problems that are way beyond anyone’s ability to touch, and you can waste a lot of time.”

Terence Tao
References

- Ivan Korec: A density estimate for the 3x + 1 problem, Mathematica Slovaka, Vol. 44(1994), 85-89
- Terence Tao: Almost all orbits of the collatz map attain almost bounded values (2019)
- Kevin Hartnett: Mathematician Proves Huge Result on ‘Dangerous’ Problem