Collatz conjecture

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History

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- Many mathematicians tried solving it.
- Soviet conspiracy aimed to slow down mathematical progress?

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Paul Erdös

Statement

Col function

$$Col(n) = \begin{cases} 3n+1, & \text{for } 2 \nmid n \\ \frac{n}{2}, & \text{for } 2 \mid n \end{cases}$$

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Collatz conjecture

For every $n \in \mathbb{N}$ we eventually reach 1 by repeatedly applying *Col* to n.

Examples

- For $n = 9$ we get the sequence:
  $9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1$

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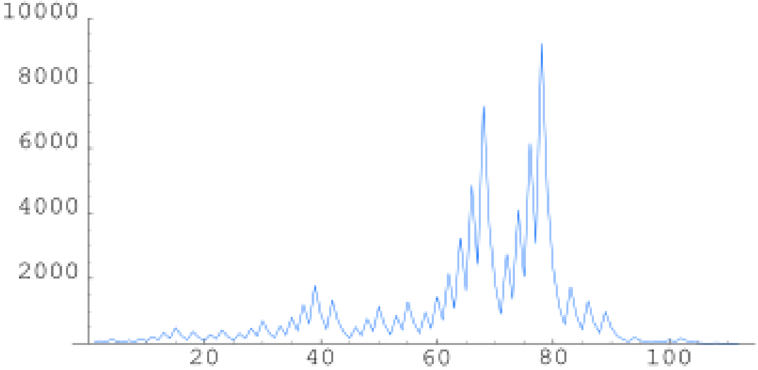
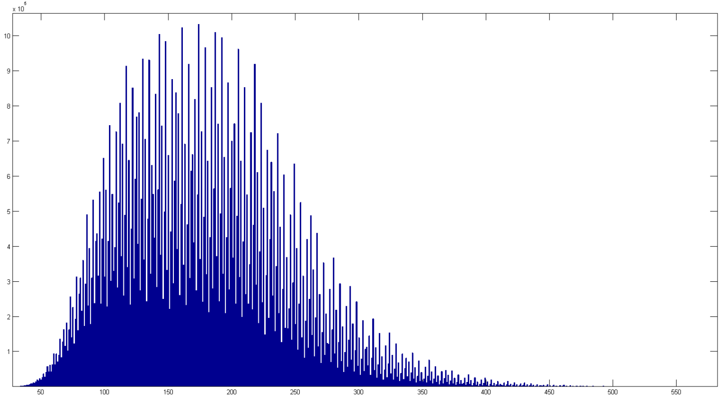
  $9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1$
- For $n = 27$ we get a sequence consisting of 111 numbers and biggest number 9232.



Highest number of steps

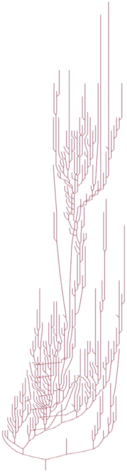
less than 10 is 9, which has 19 steps,
less than 100 is 97, which has 118 steps,
less than 1000 is 871, which has 178 steps,
less than 10^4 is 6171, which has 261 steps,
less than 10^5 is 77031, which has 350 steps,
less than 10^6 is 837799, which has 524 steps,
less than 10^7 is 8400511, which has 685 steps,
less than 10^8 is 63728127, which has 949 steps,
less than 10^9 is 670617279, which has 986 steps,
less than 10^{10} is 9780657630, which has 1132 steps,
less than 10^{11} is 75128138247, which has 1228 steps,
less than 10^{12} is 989345275647, which has 1348 steps,
less than 10^{13} is 7887663552367, which has 1563 steps,
less than 10^{14} is 80867137596217, which has 1662 steps,
less than 10^{15} is 942488749153153, which has 1862 steps,
less than 10^{16} is 7579309213675935, which has 1958 steps

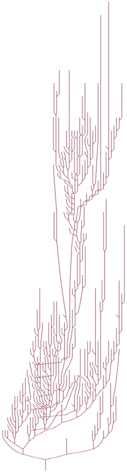
Number of steps for numbers from 1 to 10^8

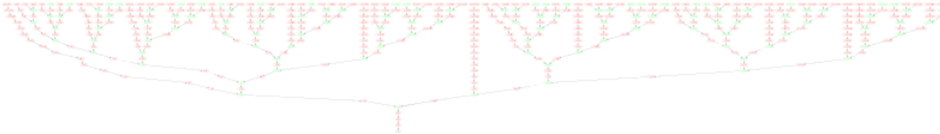


Trees


25
76
38
18
19
9
58
28
29
30
14
88
15
7
44
46
22
23
11
70
34
35
17
106
52
53
24
26
160
12
13
80
6
40
3
20
21
10
64
5
32
16
8
4
2
1







Disproving

Disproving

- finding a cycle

Disproving

- finding a cycle
- finding n, $\lim_{x \to +\infty} Col^x(n) = +\infty$

Cycles

Cycle is a sequence $(a_0, ..., a_k)$ of distinct numbers where $Col(a_i) = a_{i+1}$ for $0 < i < k$ and $Col(a_k) = a_0$.

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Eliahou Shalom (1993)

Period p of any non-trivial cycle is of the form

$$p = 301994a + 17087915b + 85137581c$$

where $b \geq 1$ and $ac = 0$

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Largest found lower bound for cycle length is 17026679261.

Supporting arguments

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- results were calculated for all $n \leq 10^{20}$

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- results were calculated for all $n \leq 10^{20}$
- on average the next odd number is $\frac{3}{4}$ of a previous one

Jeffrey C. Lagarias

"The Ultimate Challenge: The $3x + 1$ Problem"

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"Now I know lots more about the problem, and I'd say it's still impossible"

Different approach

Natural density

For $M \subseteq \mathbb{N}$ its density is defined as:

$$\lim_{x \to +\infty} \frac{card\{y \in M | y < x\}}{x}$$

Different approach

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Ivan Korec (1994)

For every $c > \log_4 3$ the set $M = \{x \in \mathbb{N} | (\exists n)(Col^n(x) < x^c)\}$ has natural density of 1.

Big breakthrough

Terence Tao

"Almost all orbits of the collatz map attain almost bounded values"

Theorem

$$Col_{min}(N) = inf_{n\in\mathbb{N}} Col^n(N)$$

Almost all orbits of the collatz map attain almost bounded values

Let $f : \mathbb{N}+1 \to \mathbb{R}$ be any function with $\lim_{N\to\infty} f(N) = +\infty$. Then one has $Col_{min}(N) < f(N)$ for almost all $N \in \mathbb{N}+1$.

How did he do it?

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- Changing density from natural to logarithmic.

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- Changing density from natural to logarithmic.
- Finding similarities to PDEs.

Partial differential equation (PDE)

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- PDEs can be used to describe a wide variety of phenomena such as sound, heat, diffusion, electrostatics, electrodynamics, fluid dynamics, elasticity, gravitation and quantum mechanics.

Similarities

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- Repeating the same process on a value to understand what happens in the future.

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- Statistical way of studying the long-term behavior of a small number of starting values.

Finding the right numbers

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Problems:

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Problems:

- numbers getting smaller

Finding the right numbers

Problems:

- numbers getting smaller
- numbers getting clumped together

Conclusions

"You could get obsessed with these big famous problems that are way beyond anyone's ability to touch, and you can waste a lot of time."

Terence Tao

References

- Ivan Korec: A density estimate for the $3x + 1$ problem, Mathematica Slovaka, Vol. 44(1994), 85-89
- Terence Tao: Almost all orbits of the collatz map attain almost bounded values (2019)
- Kevin Hartnett: Mathematician Proves Huge Result on ‘Dangerous’ Problem
- https://en.wikipedia.org/wiki/Collatz_conjecture
- https://en.wikipedia.org/wiki/Partial_differential_equation