Seymour's Second Neighbourhood Conjecture

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Overview

Intuition

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Seymour's Second Neighbourhood Conjecture suggests that in a social network described by such a graph, someone will have at least as many friends-of-friends as friends.

Statement

We are only interested in simple directed graphs

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$$d^+(v) = |N^+(v)|, d^{++}(v) = |N^{++}(v)|,$$
$$d^-(v) = |N^-(v)|, d^{--}(v) = |N^{--}(v)|,$$

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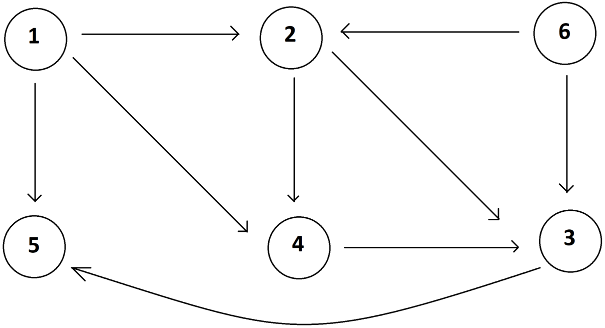
Seymour's Second Neighborhood Conjecture: Every oriented graph has a vertex with the SNP.

Example



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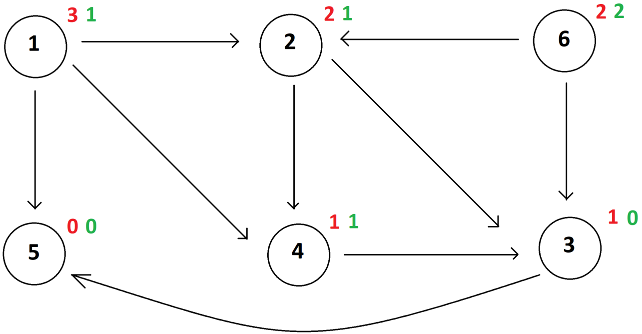

Sets

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This conjecture is actually equivalent to the SNC

Minimum out-degree

For some graphs, a vertex of minimum out-degree will be a Seymour vertex

Sinks

If a directed graph has a sink, then the sink is automatically a Seymour vertex

Sinks

In a graph without sinks, a vertex of out-degree one is always a Seymour vertex.

Triangle free graph

Let's take a triangle-free graph, and add any orientation of edges

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Any vertex v of minimum out-degree is a Seymour vertex

Tournaments

Theorem (Fisher, 1996): every tournament has a verticle with the SNP.

Random tournaments

Let T_n be a random tournament and S the set of its Seymour vertices.

$$E(|S|) = \frac{n}{2}(1 + o(1))$$

$$Var(|S|) = cn(1 + o(1))$$

λ-SNC

Chen, Shen, and Yuster: find the largest λ such that one could prove the existence of a vertex v such that

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They've proved it for value of λ equal to real root of the equation $2x^3 + x^2 - 1 = 0$ (0.6573)

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Liang, Xu: λ_m is greater than the only real root in the interval $(0, 1)$ of the polynomial:

$$2x^3 - (m-3)x^2 + (2m-4)x - (m-1)$$

(this actually implies $\lambda_3 \geq 0.6823$)

(even better) Theorem: The unique positive real root of $x^m + x^{m-1} = 1$ is a lower bound on λ_m

Directed triangle

Lemma: Let λ be the positive real constant such that every digraph contains a λ-satisfactory vertex. Then any digraph G with n vertices such that $d^+(G) \geq \frac{n}{2+\lambda}$ contains a directed triangle

Caccetta-Häggkvist Conjecture Statement

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Let $k = \frac{n}{3}$. This case is implied by Seymour's conjecture.

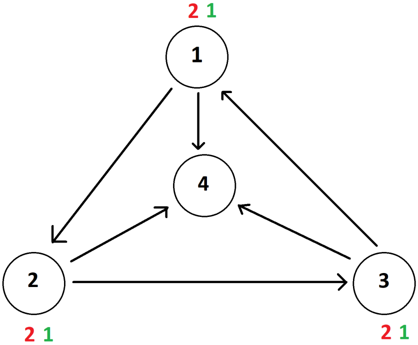
Unsuccessful Attempts

In 2007, Vishal Gupta and Erin Haller from Vale University have published proof of SSNC. However, Lemma 2 in their prove is false:

Lemma 2 (false one!) Any digraph containing at least one oriented cycle satisfies SSNC for at least one vertex on that cycle.

Link to a false proof

Counterexample



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