Lonely Runner Conjecture

Bartłomiej Jachowicz

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Abstract

One of number theory open problem is the *Lonely Runner Conjecture*. It is interesting for several reasons. First the conjecture is relatively intuitive to grasp and easy to state. This conjecture can be find in two different contexts: as a problem in Diophantine's approximation and as a geometric view obstruction problem. What is more, the difficulty of proving the Lonely Runner Conjecture may seem to increase exponentially with the number of runners. I present statement of the conjecture and known partial results.

Lonely Runner Conjecture

Definition 1.1

Consider k runners on a circular track of unit length. At time $t = 0$, all runners are at the same position and start to run. Runners have pairwise distinct speed. A runner is said to be lonely at time t if he is at a distance of at least $1/k$ from every other runner at time t.

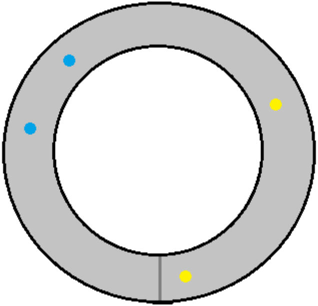
Lonely Runner Conjecture

Definition 1.2

Consider k runners on a circular track of unit length. At time $t = 0$, all runners are at the same position and start to run. Runners have pairwise distinct speed. A runner is said to be lonely at time t if he is at a distance of at least $1/k$ from every other runner at time t.

Conjecture 1.1

For any given runner there is a time t at which that runner is lonely.



Lonely Runner Conjecture

Problem 1.1

Is Lonely Runner Conjecture true for every number of runners?

History

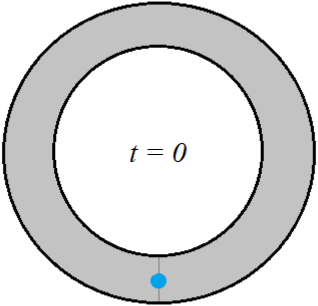
The Lonely Runner Conjecture was originally presented by J. M. Wills in 1967.

Known Results

runners	year	proved by
1	-	-
2	-	-
3	-	-
4	1972	Betke and Wills
5	1984	Cusick and Pomeranc
6	2001	Bohman, Holzman and Kleitman
7	2008	Barajas and Serra

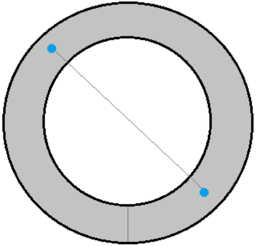
1 runner

Trivial, it works for any t, for example $t = 0$.



2 runners

Proof for LRC with 2 runners is quite simple. Due to the fact that runners speeds are not equal there is a time when runners are on opposite ends of some diameter.



4 - 7 runners

Proofs for more runner are significantly harder. The first proof of six runners was almost fifty pages, presented by a group of mathematicians from MIT. Different argument by the French mathematician Jerome Renault help to proved the case of six runners in about ten pages. More recently two Spanish mathematicians proved the case for seven runners.

Many runners

In 2011 Dubickas presented article proving the Lonely Runner Conjecture under specified assumptions.

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Theorem 2.1

Conjecture hold for $n > 16342$ *runners under assumption that the speeds of the runners are integer numbers and satisfy* $\frac{v_{j+1}}{v_j} > 1 + \frac{33 log n}{n}$ *where* $v_1 < v_2 < ... < v_n$.

Random runners are very lonely

A much stronger statement was proved for runners with random speeds.

Theorem 2.2

Let k be a fixed positive integer and let $\epsilon > 0$ be fixed real number. Let $S \subseteq \{1, 2, ..., n\}$ be a k-element subset chosen uniformly at random. Then the probability that exist time t_i when no other runner is at distance of at least $\frac{1}{2} - \epsilon$ for every runner i tends to 1 with $n \to \infty$.

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A much stronger statement was proved for runners with random speeds.

Theorem 2.3

Let k be a fixed positive integer and let $\epsilon > 0$ be fixed real number. Let $S \subseteq \{1, 2, ..., n\}$ be a k-element subset chosen uniformly at random. Then the probability that exist time t_i when no other runner is at distance of at least $\frac{1}{2} - \epsilon$ for every runner i tends to 1 with $n \to \infty$.

Proof of this theorem uses Fourier analytic methods.

The Lonely Runner Conjecture was originally formulated as a Diophantine Approximation problem.

Theorem 3.1

For all $n \in \mathbb{N}$, $\kappa(n) \geqslant \frac{1}{n+1}$

Where:

$$\kappa(n) = \underset{a \in \mathbb{R}^n}{inf}\, \lambda_n(a)$$

$$\lambda_n(a) = \underset{q \in \mathbb{Z}}{sup} \mu(q, a)$$

$$\mu(q, a) = \underset{1 \leqslant i < n}{min} \|qa_i\|$$

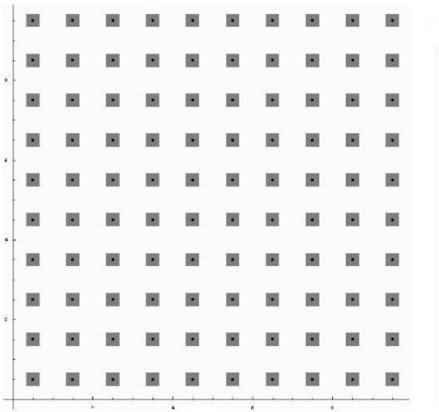
View-Obstruction

In 1971, a few years after Wills results were released, Thomas W. Cusick gave an equivalent reformulation of Lonely Runner Conjecture as a conjecture in view-obstruction.

View-obstruction

View-obstruction problems intuitively is to characterize the vector of numbers α that obstruct all rays with vector r of directions in a given dimension.

View-obstruction



Let's define this problem in a formal way:
Let E_n denote the region in R^n where all coordinates are positive, so any $x = (x_1, ..., x_n) \in E_n$ has $0 < x_i < \infty$ where $i = 1, 2, ..., n$. Suppose that C is a closed convex body in R^n and which contains the origin as an interior point.

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For each $\alpha \geqslant 0$, define αC to be the set of all $(\alpha x_1, ..., \alpha x_n)$, where $(x_1, ..., x_n)$ is a point in C; hence αC is the scale of C with the magnification of α. Define $C + (m_1, ..., m_n)$ to be the translation of C by the point $(m_1, ..., m_n) \in R^n$.

View-obstruction

Define the set of points $\Delta(C,\alpha)$ by
$\Delta(C,\alpha) = \{\alpha C + (m_1 + \frac{1}{2}, ..., m_n + \frac{1}{2}) : m_i \in \mathbb{N}, i = 1, ..., n\}$.

Problem 3.1

Find the constant $K(C)$ defined to be the lower bound of those numbers α for which every ray $r(t) = (a_1 t, ..., a_n t)$ where $a_i > 0$, $t \in [0, \infty)$, intersects $\Delta(C,\alpha)$.

Define $K(C)$ as infimum of all such α where $\Delta(C,\alpha)$ intersects every such ray and $K_n = K(C_n)$.

Theorem 3.2

The lonely runner conjecture is equivalent to finding $K(C)$.

Proof of this theorem goes by finding relation between K_n and $\kappa(n)$. Let $s = (s_1, ..., s_n) \in \mathbb{N}$ so that's is the direction of the ray $r(t) = (s_1 t, ..., s_n t)$. The closest distance, in the product metric from ray to the nearest point of $A = \{(m_1 + \frac{1}{2}, ..., m_n + \frac{1}{2}) : m_i \in \mathbb{N}\}$ is $l = \min_{t \in [0,1]} \max_{1 \leqslant i \leqslant n} \|r_i t - \frac{1}{2}\|$. Then with a few substitutions and calculations we get $K_n = 1 - 2\kappa(n)$.

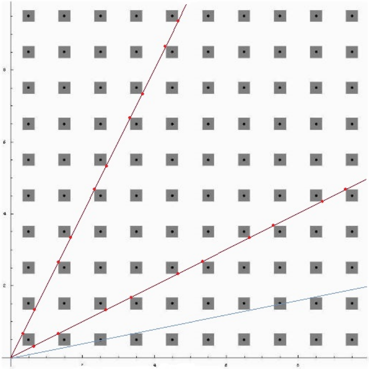
LRC for 3 runners

Theorem 4.1

$K_2 = 1/3$ *and this proves Lonely Runner Conjecture for three runners.*

LRC for 3 runners

Proof: We refer to the following picture in the proof



LRC for 3 runners

The squares are centered at half-integers and have side length of $\frac{1}{3}$, so the squares represent the set $\Delta(C_2, \frac{1}{3})$ (denote Δ_2). The two top rays have slopes of 2 and $\frac{1}{2}$ respectively while the bottom ray has a slope of $\frac{1}{5}$.

LRC for 3 runners

The rays $y = \theta x$ with $\frac{1}{2} \leqslant \theta \leqslant 2$ intersect the square with center $(\frac{1}{2},\ \frac{1}{2})$, so this slope is obstruct.

For rays with slope $\theta \in (0, \frac{1}{2})$ any of them have a chance two pass through the gap of two consecutive squares with centers $(\frac{1}{2} + n, \frac{1}{2})$, $(\frac{1}{2} + n + 1, \frac{1}{2})$. That is due to the fact we can calculate the minimal slope of a line to pass unobstructed is $\frac{1}{2}$. We can observe that on picture with line of a slope $\frac{1}{5}$.

By symmetry, Δ_2 obstructs all views with slope $\theta > 2$ if it obstructs all slopes with $\theta \in (0, \frac{1}{2})$.

LRC for 3 runners

Thus the set Δ_2 obstructs all views. From the above figure it is also evident that $\frac{1}{3}$ is the smallest number α such that $\Delta(C_2, \alpha)$ obstructs all views: for the top lines with slopes 2 and $\frac{1}{2}$ only pass through the corners of squares in Δ_2, as indicated by the points along those lines. Hence $K_2(C) = \frac{1}{3}$ and this proves the Lonely Runner Conjecture for three runners. $\square$

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- Sebastian Czerwiński, *Random numbers are very lonely*, 2012
- Open Problem Garden

The end