χ-boundedness

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Based on Paul Seymour and Alex Scott paper

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30 April 2020

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Introduction

$\chi(G)$ - chromatic number
$\omega(G)$ - clique number

Definition

*A **hole** in G is an induced cycle of length at least four, **odd hole** is one with odd length.*

Definition

*An **antihole** in G is an induced subgraph whose complement graph is a hole of complement graph $\bar{G}$ of G.*

Theorem

(Strong perfect graph theorem, 2002) *If $\chi(G) > \omega(G)$ then some induced subgraph of G is an odd hole or an odd antihole.*

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(Strong perfect graph theorem, 2002) *If $\chi(G) > \omega(G)$ then some induced subgraph of G is an odd hole or an odd antihole.*

But what if we fix some bound κ and consider graphs with $\omega(G) \leqslant \kappa$ but much larger $\chi(G)$.

Theorem

For all $\kappa \geqslant 0$, if G is a graph with $\omega(G) \leqslant \kappa$ and $\chi(G) > 2^{2^{\kappa+2}}$ then G has an odd hole.

Definition

An ***ideal*** *is a class of graphs closed under isomorphism and under induced subgraphs.*

Definition

We say that graph is ***H-free*** *if does not contain an induced subgraph isomorphic to H.*

The class of H-free graphs is an ideal; and every ideal $\mathcal{I}$ is defined by the set of (minimal) graphs H such that $\mathcal{I}$ is H-free.

Definition

*An ideal $\mathcal{I}$ is χ-**bounded** if there is a function f such that $\chi(G) \leqslant f(\omega(G))$ for each graph $G \in \mathcal{I}$. In this case we say that f is a χ-**binding** function for $\mathcal{I}$.*

Definition

*Graph H is χ-**bounding** if ideal of all H-free graphs is χ-bounded.*

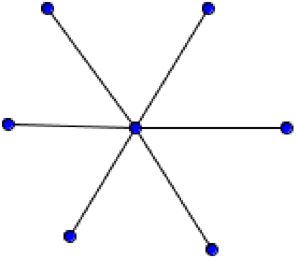
Conjecture

(The Gyárfás-Sumner conjecture) *All forests are χ-bounding*

This conjecture is easily reducible to trees, because a forest is χ-bounding if and only if all its components are χ-bounding (inductively on κ).

Theorem

Stars ($K_{1,n}$) are χ-bounding.



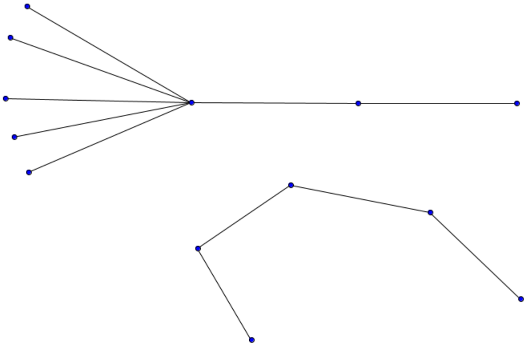
Theorem

Stars ($K_{1,n}$) are χ-bounding.

Proof: It follows from Ramsey's Theorem. Indeed, suppose $\chi(G) > R(n, \kappa)$ and $\omega(G) \leqslant \kappa$. Then G contains a vertex v of degree at least $R(n, \kappa)$ and largest clique in $N(v)$ is size at most $\kappa - 1$. This proves that there exists independent set $S \subseteq N(v)$ of size at least n. Then $\{v\} \cup S$ induces $K_{1,n}$. □

Theorem

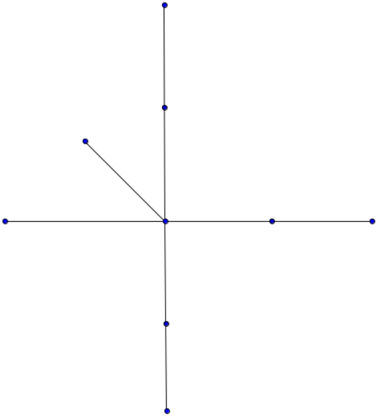
(A. Gyárfás) Paths and brooms are χ-bounding.



Original proof introduced χ-binding function $f(x) = (n-1)^{x-1}$ for P_n.

Theorem

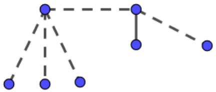
(A. Scott) Subdivisions of stars are χ-bounding.

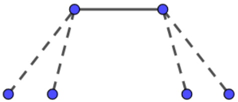


Theorem

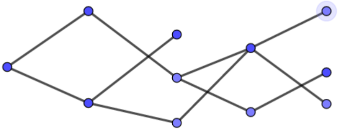
Following trees are χ-bounding:

- *trees obtained from a star and a star subdivision by adding a path joining their centres*
- *trees obtained a star subdivision by adding one vertex*
- *trees obtained from two disjoint paths by adding an edge between them*



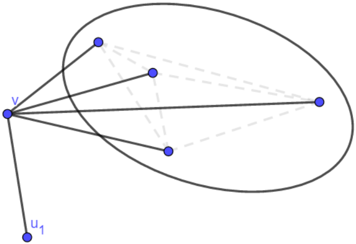


All three results are variants of following idea. First, we work by induction on clique number. Second, choose a vertex v_0 and classify all vertices by their distance from v_0 into disjoint subsets $L_0, L_1, ...$, we call this a leveling.



In this leveling, one of the levels L_k has chromatic number at least $\chi(G)/2$. Order the vertices in L_{k-1}, say $L_{k-1} = \{u_1, ..., u_m\}$ and for each i, let W_i be the set of vertices in L_k that are adjacent to u_i and nonadjacent to $u_1, ..., u_{i-1}$

This partitions L_k into the sets $W_1, W_2, ..., W_m$. Each of the W_i has bounded chromatic number (from induction), but union of all the W_i has large chromatic number. So there must exist some i and vertex $v \in W_i$ with many neighbours in $W_{i+1} \cup ... \cup W$ pairwise nonadjacent. Since u_i is adjacent to v and nonadjacent to all these neighbours, we have a little bit of a tree, that we can combine with other parts grown elsewhere.



Other ideals

We already know that graph with no odd hole is χ-bounded, but what about even holes?

Theorem

(2008) If a graph has no even hole then its chromatic number is at most twice its clique number.

Easy induction but based on complicated result that even-hole-free graph contains *bisimplicial* vertex (its neighbouring is union of two cliques).

Geometric graphs

Definition

Intersection graph: *given a collection $\mathcal{F}$ of sets, the intersection graph $I(\mathcal{F})$ has vertex set $\mathcal{F}$, and distinct $X, Y \in \mathcal{F}$ are adjacent whenever $X \cap Y$ is nonempty.*

Theorem

(A. Rok, B. Walczak) For every integer $t \geqslant 1$ the ideal of intersection graphs of curves each crossing a fixed curve in at least one and at most t points is χ-bounded.

Connections to Erdős-Hajnal conjecture

$\alpha(G)$ - size of maximal stable set

Definition

*An ideal $\mathcal{I}$ has the **Erdős-Hajnal property** if there exists some $\epsilon > 0$ such that every graph $G \in I$ has a clique or stable set of size at least $|G|^\epsilon$.*

Conjecture

(Erdős-Hajnal conjecture) *For every graph H, the ideal of H-free graphs has the Erdős-Hajnal property.*

Observation

If an ideal $\mathcal{I}$ is χ-bounded with polynomial χ-binding function f then $\mathcal{I}$ satisfies Erdős-Hajnal property.

Indeed, every graph $G \in \mathcal{I}$ satisfies

$$\alpha(G) \geqslant \frac{G}{\chi(G)} \geqslant \frac{|G|}{f(\omega(G))}$$

and so $\alpha(G)f(\omega(G)) \leqslant |G|$. If Erdős-Hajnal property would not be satisfied one could easily choose some constant ϵ based on function f to break this inequality. There is no implication in the other direction. For example, the ideal of triangle free graphs has the Erdős-Hajnal property (it's $\alpha(G) \geqslant \sqrt{n}$), but is not χ-bounded.

Definition

An ideal $\mathcal{I}$ has the ***strong Erdős-Hajnal property*** *if there exists some $\epsilon > 0$ such that for every graph $G \in I$ with $|G| > 1$ there exists disjoint $A, B \subseteq V(G)$ with $|A|, |B| \geqslant \epsilon|G|$ such that A, B are complete or anticomplete.*

Here we say that two disjoint sets A, B are ***complete*** if every vertex in A is adjacent to every vertex in B, and ***anticomplete*** if there are no edges between A, B.

Theorem

For all forests H, K, the ideal of all graphs that contain neither H nor $\bar{K}$ has the strong Erdős-Hajnal property.

Conjecture

For all forests H, the ideal of all graphs that contain neither H nor $\bar{H}$ is χ-bounded.

Bibliography

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- Maria Chudnovsky, Paul Seymour, Alex Scott *Induced subgraphs of graphs with large chromatic number.XII. Distant stars,*
- Alexandre Rok, Bartosz Walczak *Coloring curves that cross a fixed curve*, 2017