Nowhere Zero Flow and related open problems

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Outline of the seminar

- Introduction and definitions

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- Flow-colouring duality

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- The weak 3-Flow Conjecture

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- The above implies that for any bridge $e = xy, f(e, x, y) = 0$.
- Similar mental image as in flows works here as well. Pipes with water.

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- A $\mathbb{Z}$-flow f on G such that $0 < |f(\overrightarrow{e})| < k$ for all $\overrightarrow{e}$ is called a *k-flow*.
- The least k such that there exists a k-flow for G we call a *flow-number of G* and denote it as $\phi(G)$, if there is no such k then $\phi(G) = \infty$.

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- ($\rightarrow$) Color the edges with $(0,1), (1,0), (1,1)$.
- ($\leftarrow$) The colouring is exactly the value of the flow.
- It is known that NP complete to decide whether a given cubic graph is 3-colourable.

Some known facts

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- A graph has a 4-flow iff it is the union of two even graphs.
- Every 4-edge connected graph has a 4-flow.

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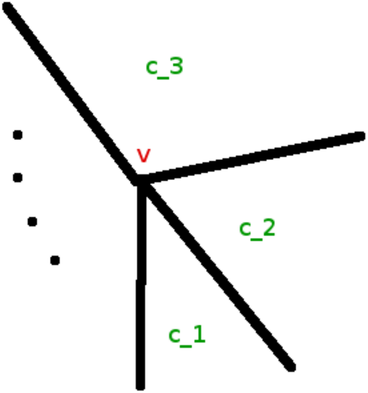
- $f(e, x, y) = c(y) - c(x) \in \{\pm 1, ..., \pm(k-1)\}$.
- Also, $f(x, V \setminus \{x\}) = 0$, so that is indeed a k-flow.

Flow-colouring duality



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- To colour the graph using flow we use a depth first search tree on G.
- We colour the root as 0 and all the other vertices as the sum of flows (in $\mathbb{Z}_k$) in the path from r to said vertex. This assures that the colours of neighboring vertices are different.

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- Every bridgeless graph has a 5-flow.
- Every bridgeless graph that does not have the Peterson graph as a minor has a 4-flow.
- However, Seymour in 1981 proved that every bridgeless graph has a 6-flow.

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- The above result is improved by Lovasz, Thomassen, Wu and Zhang for $k = 6$ in 2013.

Though Jeager's weaker conjecture has been proved

- What is the best possible k, is it the conjectured 4? Or is it the 6 that has already been proven.

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