Unique Games Conjecture

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Motivation

Showing inapproximability.

We are interested in finding best approximation ratio possible to achieve in polynomial time for NP-complete problems.
For various such problems we have approximation algorithms, however we are unable to prove that they are optimal.

Example - Min Vertex Cover

- Best approximation known: 2.
- Best inapproximability result proved: ≈ 1.36.
- Can we do better than 2?

Towards stating the UGC

Unique Label Cover (Unique Game) problem

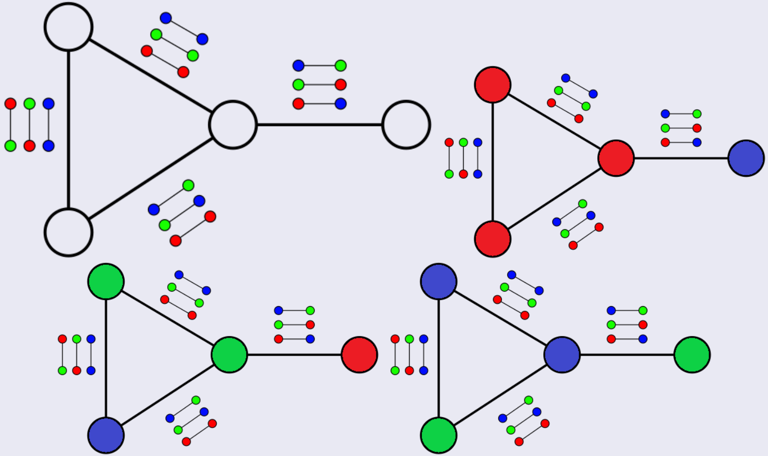
Is a CSP problem $\mathcal{U}(G(V,E),[n],\{\pi_e : e \in E\})$ defined as follows:

- $G(V,E)$ is a directed graph
- every edge e is associated with a bijection $\pi_e : [n] \to [n]$
- *labeling* $L : V \to [n]$ satisfies the constraint on edge $e = (u,v) \iff \pi_e(L(v)) = L(u)$
- the goal is to find a labeling L satisfying maximum number of constraints. A solution $OPT(\mathcal{U})$ is a best possible fraction of constraints satisfied.

Easy case - $\exists$ labeling satisfying all constraints

For each connected component of G, we can fix one vertex and try every possible assignment to it. It will automatically settle values assigned for its neighbours, and hence, for the entire connected component.

Example instance



UGC statement

What about other cases?

Given an instance $\mathcal{U}$ such that $OPT(\mathcal{U}) = 1 - \epsilon$, it seems very hard to find a solution satisfying at least ϵ fraction of constraints. This intuition is expressed as follows:

Unique Games Conjecture [Khot, 2002]

$\forall \epsilon > 0 \, \exists n = n(\epsilon)$ such that given a Unique Label Cover (Game) instance $\mathcal{U}(G(V, E), [n], \{\pi_e\})$, it is NP-hard to distinguish between these two cases:

- *YES* case: $OPT(\mathcal{U}) \geq 1 - \epsilon$
- *NO* case: $OPT(\mathcal{U}) \leq \epsilon$

Equivalently: there exists such n that given an instance $\mathcal{U} : OPT(\mathcal{U}) \geq 1 - \epsilon$, it is NP-hard to find a solution satisfying at least ϵ fraction of constraints.

UGC influence

Applications

- Inapproximability (original motive, example: Min-2SAT-Deletion problem)
- Discrete Fourier Analysis
- Geometry
- Integrality Gaps (best possible approximation ratios)
- Algorithms and Parallel Repetition

Subhash Khot, the author of the UGC, surprised by that amount of these applications, says:
"*Many of the aforementioned developments were quite unexpected (to the author at least)*"

Inapproximability results for some optimization problems

Problem	Best approx. known	Inapprox. assuming UGC	Best inapprox. known
Vertex Cover	2	$2-\epsilon$	1.36
VC on k-uniform hypergraphs, $k \geq 3$	k	$k-\epsilon$	$k-1-\epsilon$
MaxCut	0.878	$0.878+\epsilon$	$0.941+\epsilon$
Any CSP $\mathcal{C}$ with integrality gap $\alpha_{\mathcal{C}}$	$\alpha_{\mathcal{C}}$	$\alpha_{\mathcal{C}}-\epsilon$	
Max-kCSP	$O(2^k/k)$	$\Omega(2^k/k)$	$2^{k-O(\sqrt{k})}$
Max Acyclic Subgraph	2	$2-\epsilon$	$1.015-\epsilon$
Betweenness	0.333	$0.333+\epsilon$	$0,979$

Equivalent UGC formulation

Label Cover - equivalent form

The Label Cover problem can be also formulated as follows:
Suppose, we have some large $p \in \mathbb{P}$ and a finite set of variables: $\{x_1, x_2, ..., x_n\}$. We are given a system of linear equations over the integers modulo p:
$x_i - x_j \equiv c_{ij} \mod p$.
Our goal is to satisfy as many constraints as possible.

As in the previous Label Cover problem formulation, each x_i determines values of the variables that appear in one equality with x_i - every such equation represents both an edge between x_i and x_j and a bijection π, since p is prime.

Since in each equality, there appear exactly 2 different variables, a Unique Game (Label Cover) is sometimes called a "2-to-1 game".

2-to-2 Game

2-to-2 Game definition

The 2-to-2 game is a problem, where we have $p \in \mathbb{P}$, finite set of variables $\{x_1, x_2, ..., x_n\}$ and a system of linear equation over the integers, of the form:

$x_i - x_j \equiv c_{ij}, b_{ij} \mod p$

And, as we saw in 2-to-1 Game problem, we want to satisfy the greatest possible number of constraints.

Note, that each equation gives us a choice of a constant - we can either take c_{ij} or b_{ij}, so that we have 2^m possible equation sets (m is a number of equations in the problem formulation).

2-to-2 Games Theorem

Theorem statement

In, 2018, the following theorem, called "2-to-2 Games Theorem", was proved:
$\forall \epsilon > 0 \ \exists p$ such that given a $1 - \epsilon$ satisfiable 2-to-2 Game it is NP-hard to find a ϵ satisfying assignment.

Ramification

$\forall \epsilon > 0 \ \exists p$ such that given a $\frac{1}{2} - \frac{\epsilon}{2}$ satisfiable 2-to-1 Game it is NP-hard to find a ϵ satisfying assignment.

Conclusion

We are halfway there!

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