On small weak epsilon-nets for axis-parallel rectangles

Vladyslav Rachek

October 15th, 2020

Two-player game

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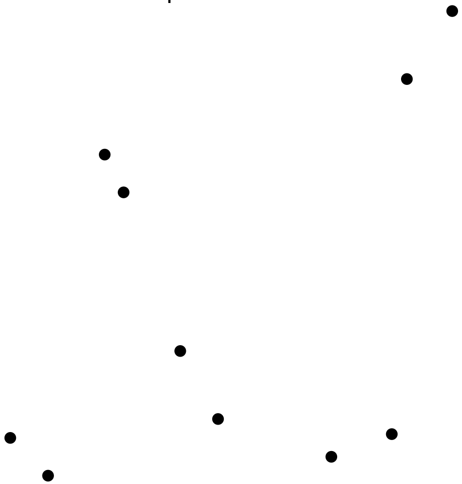
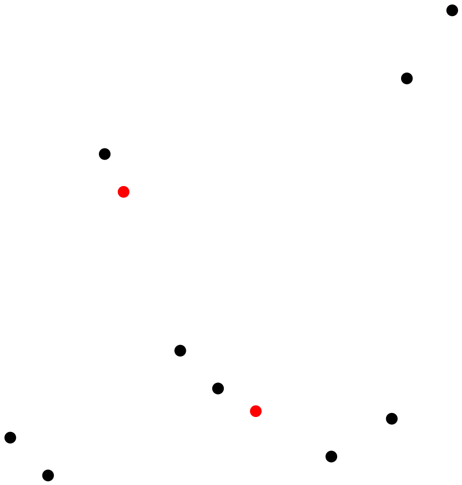
First, *Player* chooses a set P of points in **general position** in the plane where n is any number. On the picture $n = 10$:



Two-player game

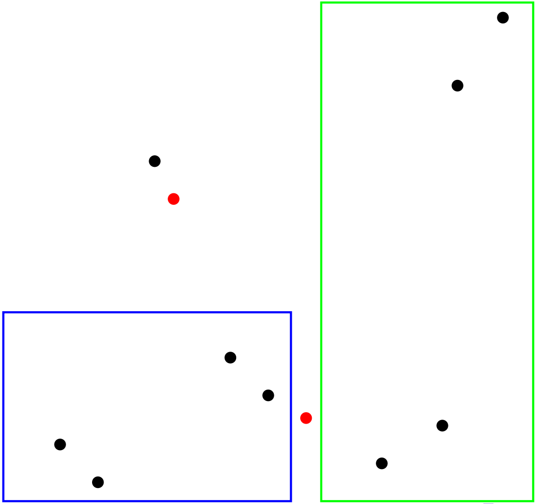
Let $\mathcal{S}$ be the set of all axis-parallel rectangles, $k = 2$.
Then *Spoiler* chooses k points on the plane and paints them red.



Two-player game

Let R be a set from $\mathcal{S}$ which contains maximal amount of only black points, and does not intersect red points.

In this game we got $|R|/n = 4/10 = 2/5$ enforced by *Spoiler*.

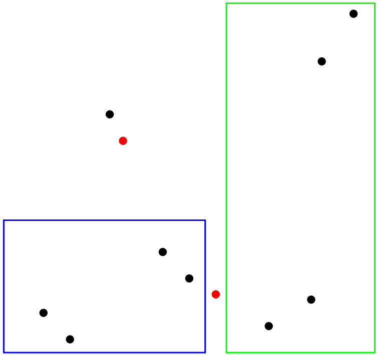


Two-player game

Let ε_k^S be the smallest number that *Spoiler* can enforce for any set of points chosen by *Player*.

Player will try to provide a construction of point set which maximizes game value.

Spoiler should provide red points for any set chosen by *Player*.



Epsilon-nets

Definition 1

Let P be an n-point set in $\mathbb{R}^2$. Consider a family $\mathcal{S}$ of sets in $\mathbb{R}^2$. A set $Q \subset \mathbb{R}^2$ is called a weak $\varepsilon -$ *net* for P with respect to $\mathcal{S}$, if for any $S \in \mathcal{S}$ with $|S \cap P| > \varepsilon n$, we have $S \cap Q = \varnothing$.

Definition 2

Let $0 \leq \varepsilon_i^{\mathcal{S}} \leq 1$ denote the smallest real number such that for any finite point set $P \subset \mathbb{R}^2$ there exist i-point set, which is $\varepsilon_i^{\mathcal{S}}$-net for P with respect to $\mathcal{S}$ ($\mathcal{S}$ is fixed).

Epsilon-nets and $\varepsilon_k^{\mathcal{R}}$

Suppose that $\varepsilon_k^{\mathcal{R}} \leq \frac{2}{k+3} \Leftrightarrow$ with k points we can restrict the largest rectangle to contain not more than $\frac{2}{k+3}n$ points for any set of n points.

What is the optimal size of an ε-net in terms of ε?

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What is the optimal size of an ε-net in terms of ε?

The best general known upper bound is $O(\frac{1}{\varepsilon} \log\log \frac{1}{\varepsilon})$

Nets for axis-parallel rectangles

Let $\mathcal{R}$ denote the family of all axis-parallel rectangles.

Theorem

$\varepsilon_1^{\mathcal{R}} = \frac{1}{2}$, $\varepsilon_2^{\mathcal{R}} = \frac{2}{5}$, $\varepsilon_3^{\mathcal{R}} = \frac{2}{6}$, $\varepsilon_4^{\mathcal{R}} \leq \frac{2}{7}$, $\varepsilon_5^{\mathcal{R}} \leq \frac{2}{8}$

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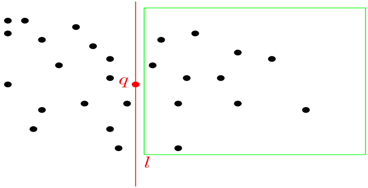
? What is the asymptotic behaviour of $\varepsilon_k^{\mathcal{R}}$?

Proof for 1-point net



Place the point q "in the middle"

Lower bound, $\varepsilon_2^{\mathcal{R}} \geq \frac{2}{5}$

Suppose $\varepsilon_2^{\mathcal{R}} < \frac{2}{5}$. For n a multiple of 5, place $\frac{n}{5}$ points in each of the rectangles $A_{1,1}, A_{1,3}, A_{2,2}, A_{3,1}, A_{3,3}$. It follows that $\varepsilon_2^{\mathcal{R}} \geq \frac{2}{5}$.

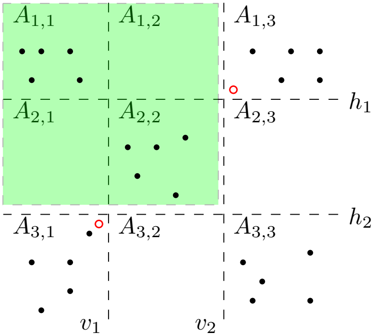
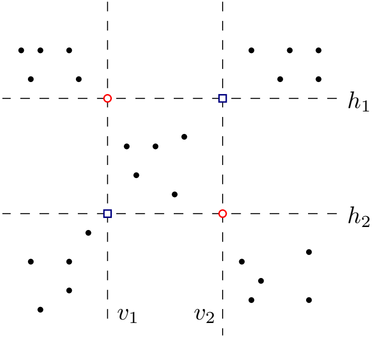


Figure: Red circles are points of Q, green rectangle contains $\frac{2n}{5}$ points and avoids Q.

Upper bound, $\varepsilon_2^{\mathcal{R}} \leq \frac{2}{5}$



Suppose we are given a set P of n points where n is a multiple of 5.

One of the sets $Q_1 = \{h_1 \cap v_1, h_2 \cap v_2\}$ and $Q_2 = \{h_1 \cap v_2, h_2 \cap v_1\}$ is a $\frac{2}{5}$-net for P.

For a contradiction, suppose that neither Q_1 nor Q_2 is a $\frac{2}{5}$-net for P.

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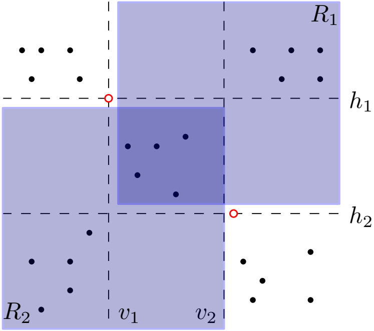
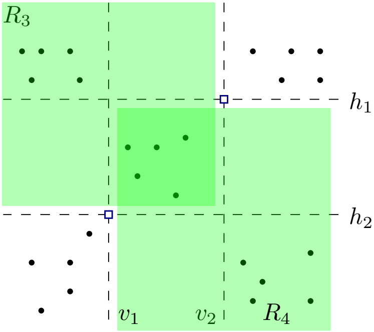


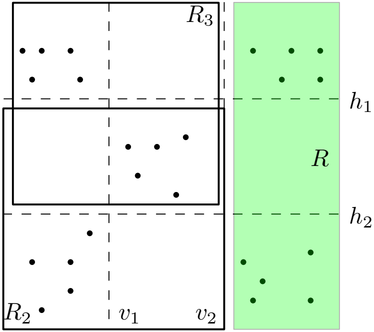
Since Q_1 is not a $\frac{2}{5}$-net for P, at least one of the rectangles R_1 and R_2 contains more than $\frac{2n}{5}$ points of P.

Upper bound, $\varepsilon_2^{\mathcal{R}} \leq \frac{2}{5}$



Similarly, because Q_2 is not a $\frac{2}{5}$-net for P, at least one of the rectangles R_3 and R_4 contains more than $\frac{2n}{5}$ points of P.

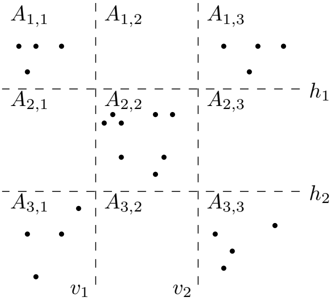
Upper bound, $\varepsilon_2^{\mathcal{R}} \leq \frac{2}{5}$



For the sake of the argument, suppose that R_2 and R_3 each contain at least $2n/5$ points of P.

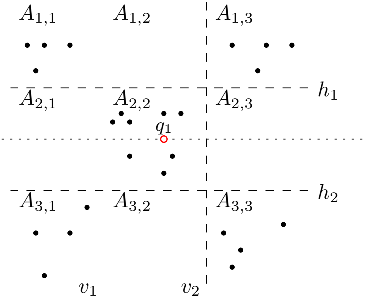
It follows that R, R_2 and R_3 altogether contain strictly more than $\frac{2n}{5} + (\frac{2n}{5} + \frac{2n}{5} - \frac{n}{5}) = n$ points, a contradiction.

Lower bound, $\varepsilon_3^{\mathcal{R}} \geq \frac{2}{6}$



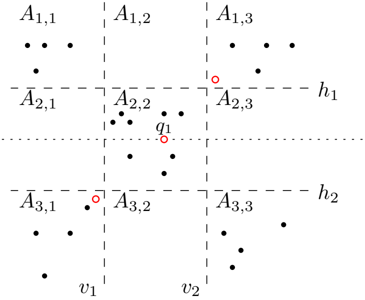
Suppose that $\varepsilon_3^{\mathcal{R}} < \frac{2}{6}$.
Let P be the set of n points where n is a multiple of 6, defined as follows: place $\frac{n}{6}$ points inside each of the four rectangles $A_{1,1}, A_{1,3}, A_{3,1}, A_{3,3}$, and place the remaining $\frac{2n}{6}$ points inside $A_{2,2}$.

Lower bound, $\varepsilon_3^{\mathcal{R}} \geq \frac{2}{6}$



Suppose there exists a 3-point $\varepsilon_3^{\mathcal{R}}$-net Q for P. Since $\varepsilon_3^{\mathcal{R}} < \frac{2}{6}$, one of the points of the net, say $q_1 \in Q$ should lie inside $A_{2,2}$.

Lower bound, $\varepsilon_3^{\mathcal{R}} \geq \frac{2}{6}$



Furthermore, because each of the outer strips contains exactly $\frac{2n}{6}$ points, the net Q has at least one point inside each of those strips.

Since q_1 is already in $A_{2,2}$, it follows that the remaining two points of Q are placed in $A_{1,1}$ and $A_{3,3}$ or in $A_{1,3}$ and $A_{3,1}$. Assume the latter w.l.o.g.

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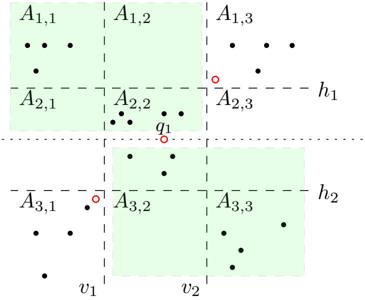
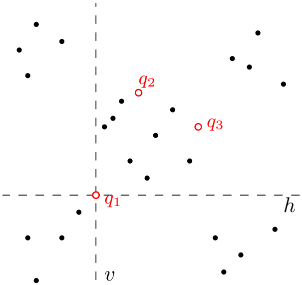


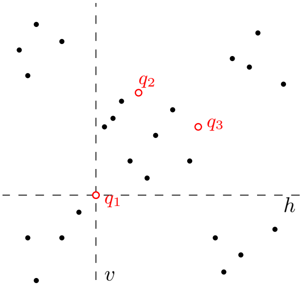
Since $\frac{2n}{6}$ is even, either above or below the horizontal line defined by q_1 there are at least $\frac{n}{6}$ points from $A_{2,2}$. That way, at least one of two rectangles contains no fewer than $2n/6$ points of P and avoids Q, a contradiction.

Upper bound, $\varepsilon_3^{\mathcal{R}} \leq \frac{2}{6}$



Let P be a set of n points where n is divisible by 6. Choose $q_1 \in Q$ so that the vertical line v passing through q_1 has exactly $n/3$ points of P on its left, and the horizontal line h passing through q_1 has exactly $n/3$ points of P below.

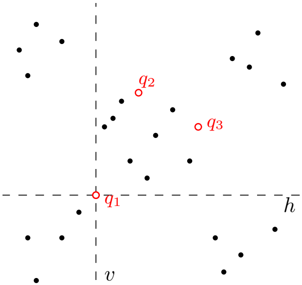
Upper bound, $\varepsilon_3^{\mathcal{R}} \leq \frac{2}{6}$



Next, let P_{above} be the set of all points of P above h, and P_{right} be the set of all points of P to the right of v.

Let q_2 be a point which forms a $\frac{1}{2}$-net for P_{above}, and q_3 — a point which forms a $\frac{1}{2}$-net for P_{right}.

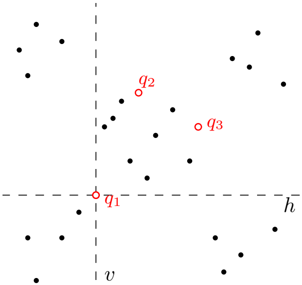
Upper bound, $\varepsilon_3^{\mathcal{R}} \leq \frac{2}{6}$



Every rectangle avoiding Q avoids q_1 and thus lies entirely inside one of the four open half-planes determined by h or v.

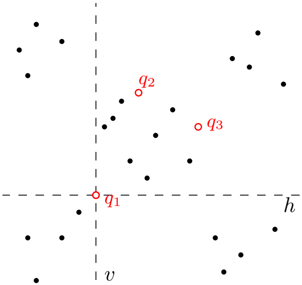
Clearly, any rectangle lying fully below h or fully to the left of v contains at most $\frac{2n}{6}$ points, due to the definition of these lines.

Upper bound, $\varepsilon_3^{\mathcal{R}} \leq \frac{2}{6}$



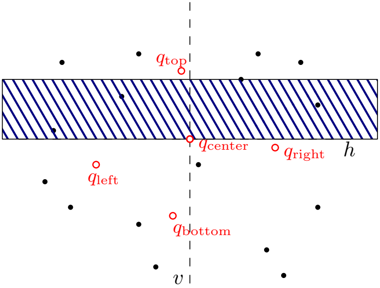
Any other rectangle which avoids q_1 must lie either fully above h or fully to the right of v, hence the set of points it contains is a subset of P_{above} or a subset of P_{right}, respectively.

Upper bound, $\varepsilon_3^{\mathcal{R}} \leq \frac{2}{6}$



Since q_2 and q_3 are $\frac{1}{2}$-nets for P_{above} and P_{right}, respectively, no such rectangle can contain more than $\frac{2n}{3} \cdot \frac{1}{2} = \frac{n}{3}$ points of P, which shows that $Q = \{q_1, q_2, q_3\}$ is a $\frac{2}{6}$-net for P.

Upper bound $\varepsilon_5^{\mathcal{R}} \leq \frac{2}{8}$

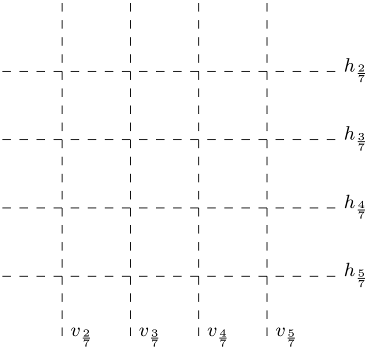


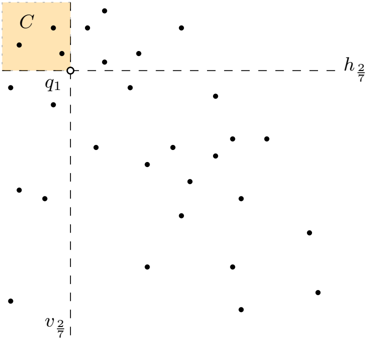
Let P be a set of n points where $8|n$.
Let $q_{\text{center}} = h \cap v$.
Let $Q = \{q_{\text{center}}, q_{\text{left}}, q_{\text{right}}, q_{\text{top}}, q_{\text{bottom}}\}$.
Each of the sets $P_{\text{left}}, P_{\text{right}}, P_{\text{top}}, P_{\text{bottom}}$ contains at most $\frac{n}{2}$ points and if any rectangle R avoids Q, then the set of points it contains is a subset of one of the sets $P_{\text{left}}, \ldots, P_{\text{bottom}}$. Since Q contains a $\frac{1}{2}$-net for each of the latter sets, any such rectangle contains at most $\frac{1}{2} \cdot \frac{n}{2} = \frac{n}{4}$ points of P.
Therefore our set Q is a $\frac{1}{4}$-net for P.

$\varepsilon_4^{\mathcal{R}} \leq \frac{2}{7}$ – lemma 1

Lemma 1

If one of the corners determined by the four lines $v_{\frac{2}{7}}, v_{\frac{5}{7}}, h_{\frac{2}{7}}$ and $h_{\frac{5}{7}}$ contains at least $\frac{n}{7}$ points of P, then there exists a 4-point $\frac{2}{7}$-net for P.

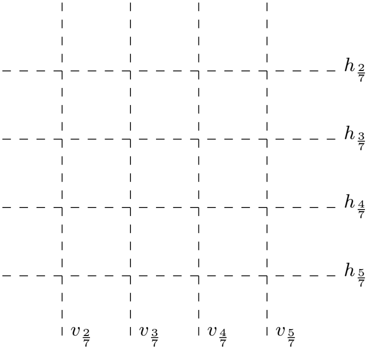


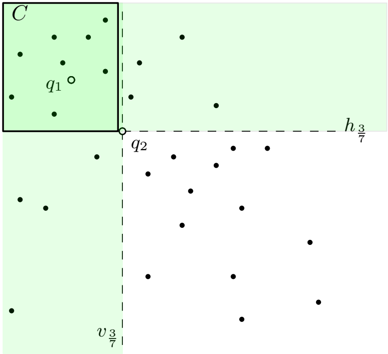


$\varepsilon_4^{\mathcal{R}} \leq \frac{2}{7}$ – lemma 2

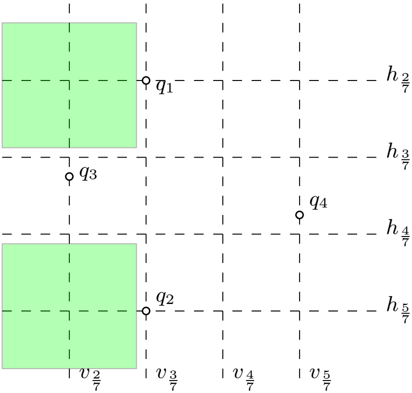
Lemma 2

If one of the corners determined by the four lines $v_{\frac{3}{7}}, v_{\frac{4}{7}}, h_{\frac{3}{7}}$ and $h_{\frac{4}{7}}$ contains at least $\frac{2n}{7}$ points of P, then there exists a 4-point $\frac{2}{7}$-net for P.





Proof for the first case, $\varepsilon_7^{\mathcal{R}} \leq \frac{2}{7}$



Suppose that inside middle 3×3 rectangle there are at mose $2n/7$ points.

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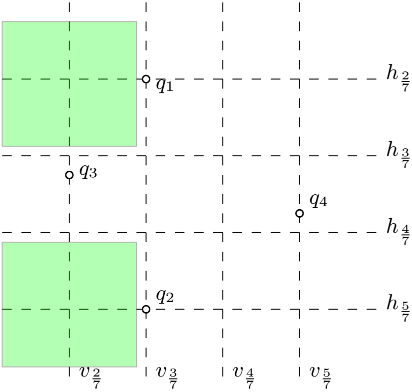

Note that there are exactly $3n/7$ points to the left of $v_{\frac{3}{7}}$, which passes through q_1 and q_2.

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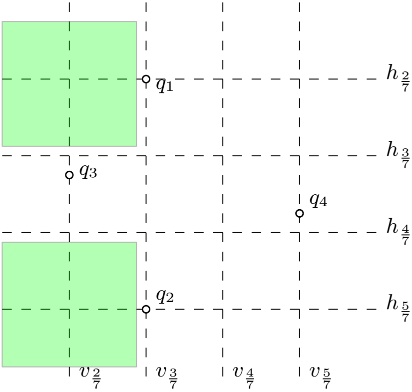

Note that there are exactly $3n/7$ points to the left of $v_{\frac{3}{7}}$, which passes through q_1 and q_2.

Therefore we can choose the third point of a net q_3 in a way that any rectangle to the left of $v_{\frac{3}{7}}$ and above or below q_3 contains at most $2n/7$ points.

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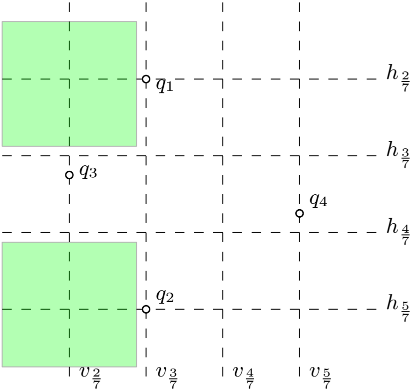

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The point q_4 is taken to be a $\frac{1}{2}$-net for those points of P which lie to the right of $v_{\frac{3}{7}}$.

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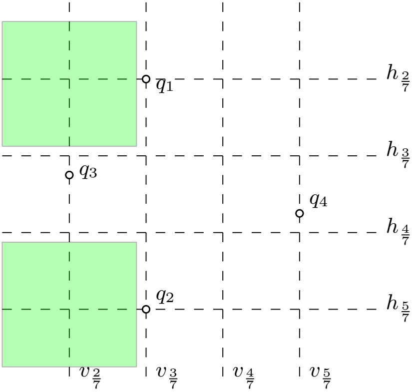

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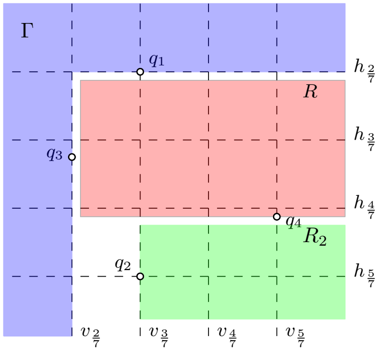
QED?

Proof for the first case, $\varepsilon_7^{\mathcal{R}} \leq \frac{2}{7}$

Can there exist R containing more than $\frac{2}{7}$-fraction of points?



R lies between $h_{\frac{2}{7}}$ and $h_{\frac{5}{7}}$.

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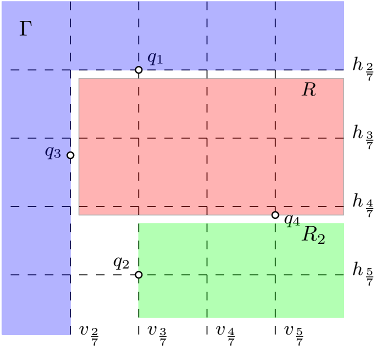

R lies between $h_{\frac{2}{7}}$ and $h_{\frac{5}{7}}$.
R cannot lie inside inside $M_{3\times 3}$.

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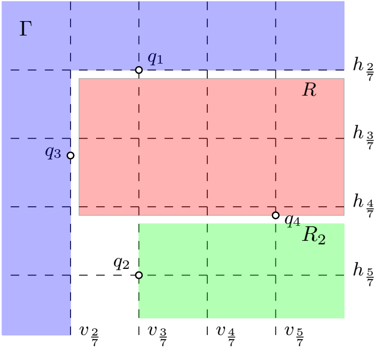

R lies between $h_{\frac{2}{7}}$ and $h_{\frac{5}{7}}$.
R cannot lie inside inside $M_{3\times 3}$.
It follows that R lies above or below q_4, almost touches $v_{\frac{2}{7}}$ with its left side and crosses $v_{\frac{5}{7}}$ with its right side.

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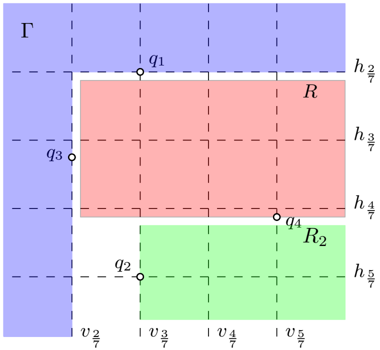

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R cannot lie inside inside $M_{3\times3}$.

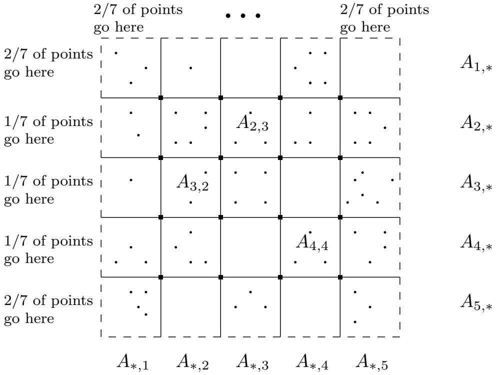
It follows that R lies above or below q_4, almost touches $v_{\frac{2}{7}}$ with its left side and crosses $v_{\frac{5}{7}}$ with its right side.

Observe that there are at most $n/7$ points of P in the left upper corner.

Grid conjecture, grid conditions

Conjecture (Walczak, Langerman, personal communication)

Let $k \in \mathbb{N}$ and P be a point set of size n where n is divisible by $k+3$. There exists a k-point $\frac{2}{k+3}$-net for P with respect to $\mathcal{R}$, where the net is chosen from the set of grid points.



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- When the conjecture is false?
  If and only if there exists a point set P of size n, meeting grid conditions, such that for each k-point set Q chosen from the set of grid points there exists a rectangle which contains strictly more than $\frac{2n}{k+3}$ points and avoids Q.

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- Instead of R we can consider an open rectangle R', which contains all points from all cells which R intersects and is stretched to the boundary of these cells.

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- Instead of R we can consider an open rectangle R', which contains all points from all cells which R intersects and is stretched to the boundary of these cells.

$\Rightarrow$ It suffices to consider a finite amount of rectangles!

Not all rectangles are worth considering

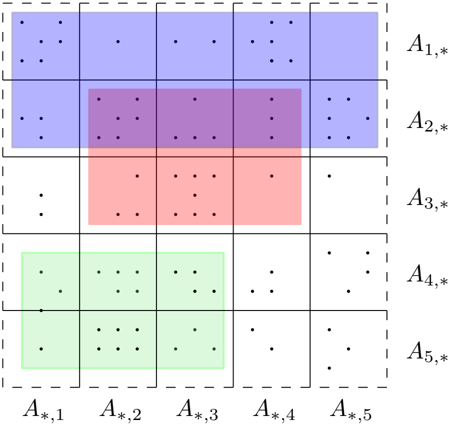


Figure: Top-down: oversufficient, insufficient and sufficient rectangles respectively ($k = 4$).

We will denote the set of all sufficient rectangles by R_{suf} and the set of all oversufficient rectangles by R_{oversuf}.

Make our observations programmable

Observation

Any $\frac{2}{k+3}$-net must hit all oversufficient rectangles.

We denote the set of all such nets by $\mathcal{Q}$.

Observation

A counterexample to conjecture exists if and only if there exists a subset $S \subseteq R_{\text{suf}}$ such that the following conditions are met:

(i) *No net from $\mathcal{Q}$ hits all rectangles from S (for each net $Q \in \mathcal{Q}$ there exist some rectangle $R \in S$ which avoids Q).*

(ii) *There exist a point set P meeting grid conditions such that every rectangle in S contains strictly more than $\frac{2}{k+3}$-fraction of points of P.*

Naive algorithm

procedure Find-set
  for $S \subseteq R_{\mathrm{suf}}$ **do** ▷ $O(2^{|R_{\mathrm{suf}}|})$ iterations
    if $\forall Q \in \mathcal{Q} \;\; \exists R \in S$ such that $R \cap Q = \varnothing$ **then** ▷ no net can hit all rectangles from S
      if $\exists P \; : \; \forall R \in S \;\; R$ contains more than the $\frac{2}{k+3}$-fraction of points of P **then**
        return P ▷ P is a counterexample
      end if
    end if
  end for
  return $\varnothing$
end procedure

Enter linear programming

How to look for a point set P?

Enter linear programming

Matrix $X = (x_{i,j})_{1 \le i,j \le k+1}$ satisfies:

(P1) $x_{i,j} \ge 0$ for any i, j

(P2) $\sum_{i,j} x_{i,j} = 1$

(P3) $\sum_j x_{1,j} = \sum_j x_{k+1,j} = \sum_i x_{i,1} = \sum_i x_{i,k+1} = \frac{2}{k+3}$, and
$\sum_i x_{i,h} = \sum_j x_{h,j} = \frac{1}{k+3}$ for $2 \le h \le k$

(P1, P2) basically say that X represents some set of points P
(P3) forces the grid conditions on any set which X represents.

Enter linear programming

Now it is easy to add another condition. Fix some $S \subseteq R_{\mathrm{suf}}$. For any $R \in S$, let $C_R := \{(i,j) : R \text{ intersects the } (i,j)\text{-cell of a grid}\}$. Now, let

$(P4(S))$ For all $R \in S$ $\sum\limits_{i,j \in C_R} x_{i,j} > \frac{2}{k+3}$

Enter linear programming

```
procedure Find-set
    for S ⊆ R_suf do                                          ▷ O(2^{|R_suf|}) iterations
        if ∀Q ∈ 𝒬 ∃R ∈ S such that R ∩ Q = ∅ then      ▷ no net can hit all rectangles from S
            if ∃X : X satisfies (P1-P3) and (P4(S)) then
                return X                        ▷ X encodes some counterexample
            end if
        end if
    end for
    return ∅
end procedure
```

Final touch

Observation

Take any $S_1, S_2 \in \mathcal{P}$ such that $S_1 \subset S_2$. If $LP(S_2)$ has a solution, then $LP(S_1)$ has a solution.

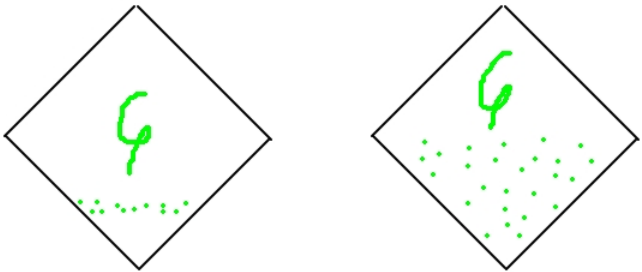
Let $\mathcal{G} \subset \mathcal{P}$ denote the set of all inclusion-minimal elements of $\mathcal{P}$:

$$\mathcal{G} := \{S \in \mathcal{P} \;:\; \nexists\, S' \in \mathcal{P} \text{ such that } S' \subset S\}$$

Observation

For any $S \in \mathcal{G}$, S cannot contain two rectangles R_1 and R_2 such that R_1 lies fully inside R_2. Thus, S must form an antichain in the set $\mathcal{S}$ ordered by inclusion (in a geometrical sense).

Check your intuition!



Harvest the ~~fruits~~ counterexamples

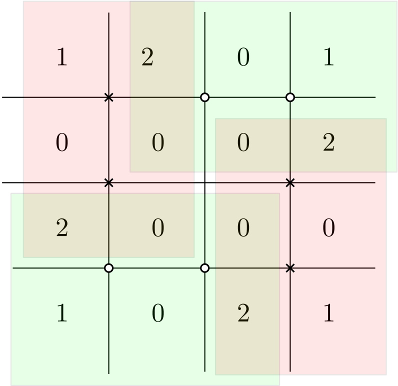


Figure: Four presented rectangles contain more than $\frac{2}{6} \cdot 12$ points each.

Harvest the ~~fruits~~ counterexamples

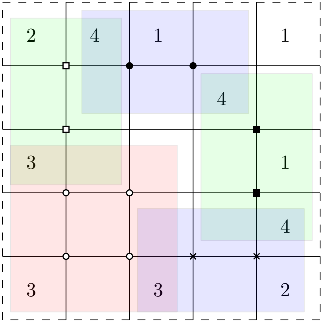


Figure: Five presented rectangles contain more than $\frac{2}{7} \cdot 28$ points each.

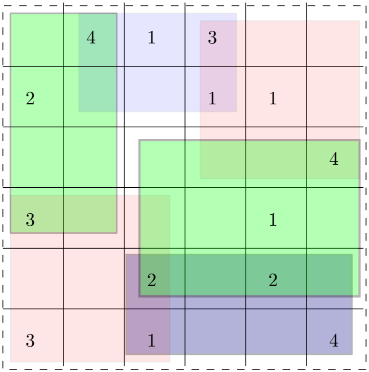


Figure: Six presented rectangles contain more than $\frac{2}{8} \cdot 32$ points each

Bringing upper bound back

It was claimed that $\varepsilon_6^{\mathcal{R}} \le \frac{2}{9}$, and the proof given there was via a program which chooses a net from the set of grid points. Our result shows that that proof was incorrect, so whether $\varepsilon_6^{\mathcal{R}} \le \frac{2}{9}$ remains unsolved.

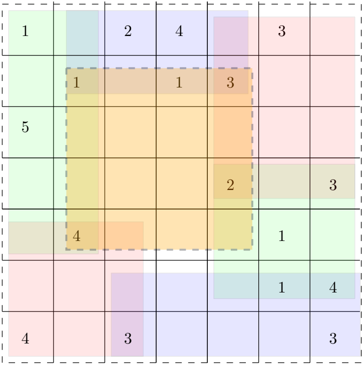


Figure: Seven presented rectangles contain more than $\frac{2}{9} \cdot 45$ points each

Open problems

- True value of $\varepsilon_6^{\mathcal{R}}$
- Asymptotic behaviour of $\varepsilon_i^{\mathcal{R}}$