# The Erdős-Hajnal Conjecture

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Theoretical Computer Science

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# Sources

1. Open Problem Garden
2. The Erdös-Hajnal Conjecture - A Survey (2016) by M. Chudnovsky
3. Ramsey-type theorems (1989) by P. Erdős, A. Hajnal
4. A Ramsey-type theorem for bipartite graphs (2000) by P. Erdős, A. Hajnal, J. Pach

# Large cliques and independent sets

## All graphs

For all sufficiently large $n$:

- every $n$-vertex graph $G$ has either a clique or independent set of size $\frac{\log n}{2 \log 2}$;
- there exists an $n$-vertex graph $G$ not containing a clique or independent set of size $\frac{2 \log n}{\log 2}$.

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## $H$-free graphs (Erdős, Hajnal 1989)

For every graph $H$, there exists a constant $c(H) > 0$ s.t. every $H$-free graph $G$ with $n$ vertices has either a clique or independent set of size at least $e^{c(H)\sqrt{\log n}}$.

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## The Erdős-Hajnal conjecture (1989)

For every graph $H$, there exists a constant $\delta(H) > 0$ s.t. every $H$-free graph $G$ with $n$ vertices has either a clique or independent set of size at least $n^{\delta(H)}$.

# Notation

## Definitions

- $G^c$ = complement of graph $G$
- $\omega(G)$ = the maximum size of a clique in $G$
- $\alpha(G)$ = the maximum size of an independent set in $G$
- $\chi(G)$ = the chromatic number of $G$

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## The Erdős-Hajnal conjecture

For every graph $H$, there exists a constant $\delta(H) > 0$ s.t. every $H$-free graph $G$ with $n$ vertices has $\omega(G) \geq n^{\delta(H)}$ or $\alpha(G) \geq n^{\delta(H)}$.

# Perfect graphs

**Perfect graph**

Graph $G$ is perfect iff $\omega(H) = \chi(H)$ for every induced subgraph $H$ of $G$.

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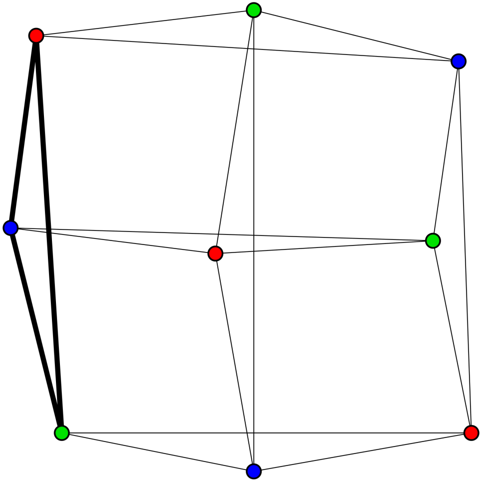
## Strong Perfect Graph Theorem (Chudnovsky et al. 2006)

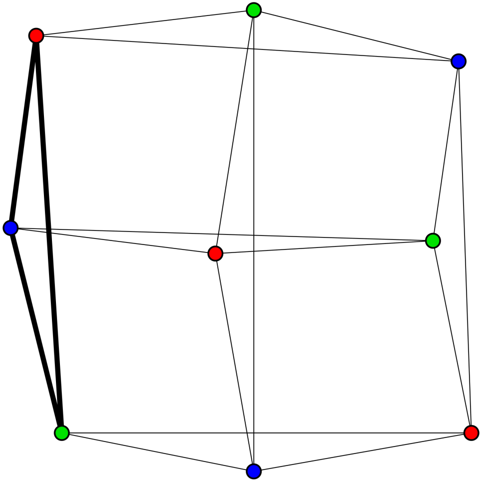
Graph $G$ is perfect iff no induced subgraph of $G$ or $G^c$ is an odd cycle of length at least 5.

# Perfect graphs



Source: Wikipedia

# Perfect graphs



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$$|V(G)| \leq \chi(G)\alpha(G)$$

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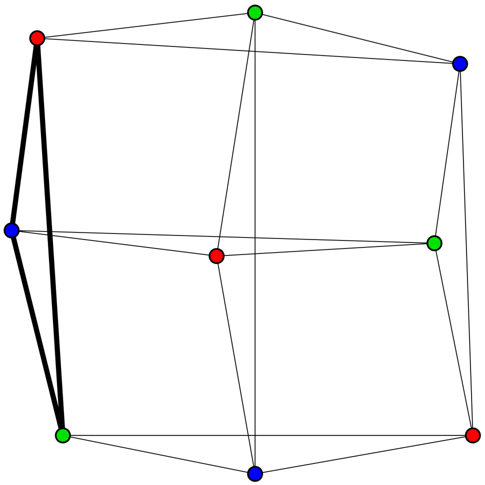

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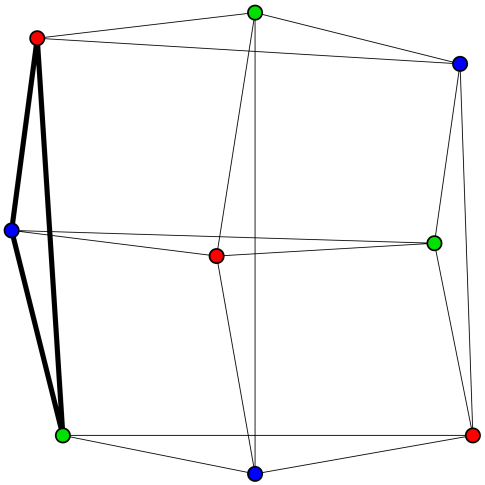

Source: Wikipedia

$$|V(G)| \leq \chi(G)\alpha(G) = \omega(G)\alpha(G)$$
$$\Longrightarrow$$
$$\omega(G) \geq \sqrt{|V(G)|} \text{ or } \alpha(G) \geq \sqrt{|V(G)|}$$

**Theorem**

If $G$ is a perfect graph with $n$ vertices, then $\omega(G) \geq \sqrt{n}$ or $\alpha(G) \geq \sqrt{n}$.

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**Conjecture equivalent to Erdős-Hajnal**

For every graph $H$, there exists a constant $\psi(H) > 0$ s.t. every $H$-free graph $G$ with $n$ vertices has a perfect induced subgraph with at least $n^{\psi(H)}$ vertices.

# Partial results

**The Erdős-Hajnal property**

Graph $H$ has the Erdős-Hajnal property if and only if there exists a constant $\delta(H) > 0$ s.t. every $H$-free graph $G$ with $n$ vertices has $\omega(G) \geq n^{\delta(H)}$ or $\alpha(G) \geq n^{\delta(H)}$.

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## The Erdős-Hajnal conjecture

Every graph $H$ has the Erdős-Hajnal property.

# Partial results

## Graphs known to have the Erdős-Hajnal property

- complete graphs and their complements
- all graphs with at most 4 vertices
- graphs formed by ”substitution” operation (Alon, Pach, Solymosi 2001)
- the bull graph (Chudnovsky, Safra 2008)

# Partial results

## Graphs known to have the Erdős-Hajnal property

- complete graphs and their complements
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- the bull graph (Chudnovsky, Safra 2008)

## Open question

Is the Erdős-Hajnal conjecture true when $H \cong C_5$?

# Complete graphs and their complements

## Ramsey numbers

- $R(n, m)$ is the minimum $k$ s.t. every graph $G$ with $k$ vertices has $\omega(G) \geq n$ or $\alpha(G) \geq m$
- $R(n, m) \leq \binom{n+m-2}{n-1}$

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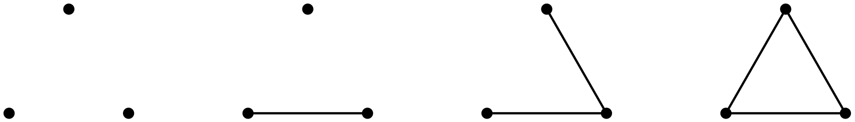
## Forbidden $K_n$ graph

- A $K_n$-free graph on $f_n(m) = R(n, m)$ vertices has an independent set of size $m$
- Function $f_n(x)$ grows polynomially
- $\alpha(G)$ grows polynomially

# Small graphs

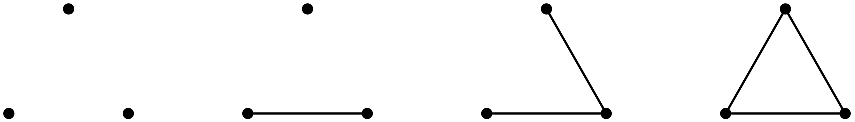
## Forbidden graphs with 3 vertices



# Small graphs

## Forbidden graphs with 3 vertices

- $K_3$ is a clique



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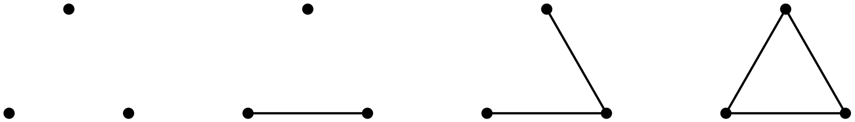
## Forbidden graphs with 3 vertices

- $K_3$ is a clique
- $P_3$-free graph is a disjoint set of cliques, so $\omega(G) \geq \sqrt{|V(G)|}$ or $\alpha(G) \geq \sqrt{|V(G)|}$



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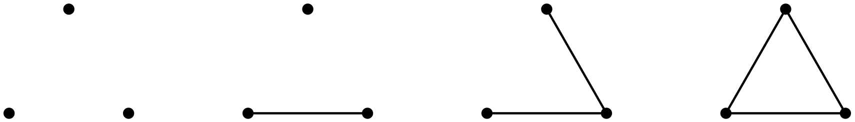
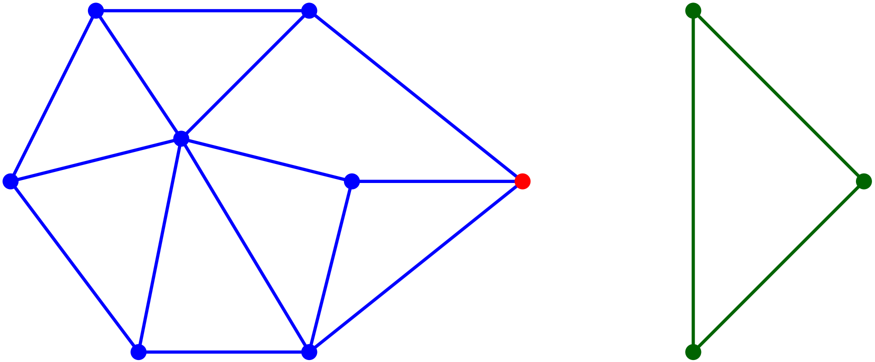
- $K_3$ is a clique
- $P_3$-free graph is a disjoint set of cliques, so $\omega(G) \geq \sqrt{|V(G)|}$ or $\alpha(G) \geq \sqrt{|V(G)|}$
- The remaining 3-vertex graphs are complements of $K_3$ and $P_3$



# Substitution



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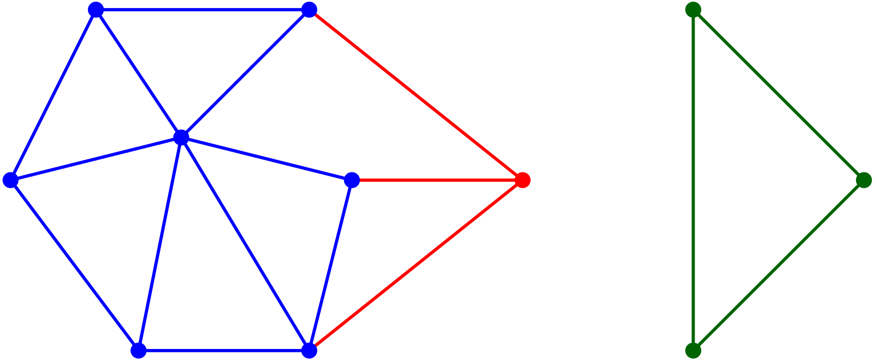

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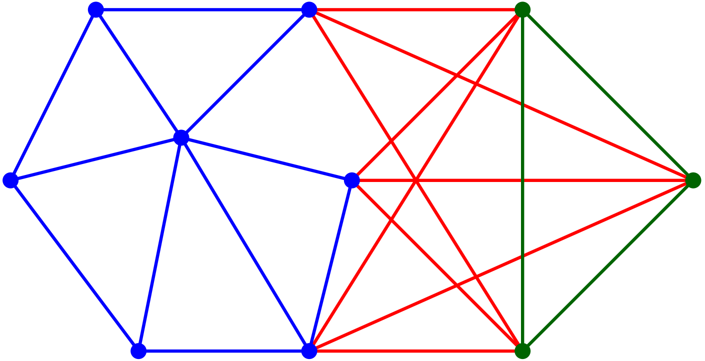

# Substitution

## Substitution operation

Given graphs $H_1$ and $H_2$, on disjoint vertex sets, each with at least two vertices, and $v \in V(H_1)$, we say that $H$ is obtained from $H_1$ by subtituting $H_2$ for $v$ if:

- $V(H) = (V(H_1) \cup V(H_2)) \setminus \{v\}$
- $H[V(H_2)] = H_2$
- $H[V(H_1) \setminus \{v\}] = H_1 \setminus v$
- for $u \in V(H_1), w \in V(H_2)$: $uw \in E(H) \iff uv \in E(H_1)$

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## Prime graph

A graph is prime if it cannot be obtained from smaller graphs by substitution.

# Substitution

**Theorem (Alon, Pach, Solymosi 2001)**

If $H_1$ and $H_2$ are graphs with the Erdős-Hajnal property, and $H$ is obtained from $H_1$ and $H_2$ by subtitution, then $H$ has the Erdős-Hajnal property.

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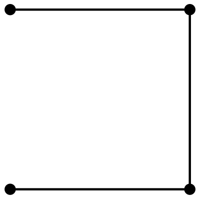
## Proof idea

- Let $G$ be an $H$-free graph with $n$ vertices without "large" cliques and independent sets
- Every induced subgraph of $G$ with $n^{\varepsilon}$ vertices contains an induced copy of $H_1$ and $H_2$
- By counting, some copy of $H_1 \setminus v$ can be extended to $H_1$ in at least $n^{\varepsilon}$ ways
- There is $H_2$ among possible extensions, so $G$ is not $H$-free

## Forbidden graphs with 4 vertices

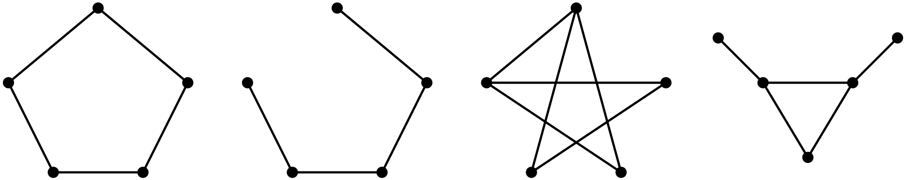
- We only need to consider prime graphs
- There is only one 4-vertex prime graph: $P_4$
- $P_4$-free graphs are perfect, so $P_4$ has the Erdős-Hajnal property



# Small graphs

## Prime graphs with 5 vertices

- cycle $C_5$ - open...
- path $P_5$ - open...
- complement of $P_5$ - open...
- the bull graph - **solved** (Chudnovsky, Safra 2008)



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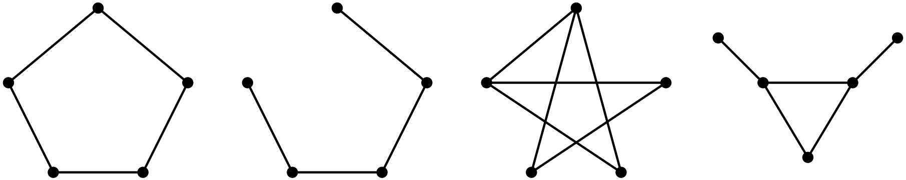
## Theorem (Chudnovsky, Safra 2008)

Every bull-free graph $G$ with $n$ vertices has $\omega(G) \geq n^{1/4}$ or $\alpha(G) \geq n^{1/4}$.



# A weaker conjecture

## Conjecture

For every graph $H$, there exists a constant $\epsilon(H) > 0$ s.t. every $\{H, H^c\}$-free graph $G$ with $n$ vertices has either a clique or independent set of size at least $n^{\epsilon(H)}$.

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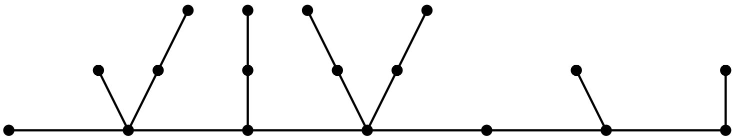
## History of solved cases of $H$

- 5-edge path (Chudnovsky, Seymour 2013)
- all paths (Bousquet, Lagoutte, Thomassé 2014)
- hooks (Bousquet, Lagoutte, Thomassé 2014)
- caterpillars (Liebenau, Pilipczuk 2017)
- caterpillar subdivisions (Liebenau, Pilipczuk, Seymour, Spirkl 2018)



# Other classes of graphs

**Polynomially bounded $\max(\omega(G), \alpha(G))$**

- string graphs (Tomon 2020)
- graphs with no induced "holes with hats" (Chudnovsky, Seymour 2020)

# Bipartite variant

## Definitions

Let $A, B \subseteq V(G)$ be disjoint sets of vertices.

- $A$ is complete to $B$ iff every vertex of $A$ is adjacent to every vertex of $B$
- $A$ is anticomplete to $B$ iff every vertex of $A$ is non-adjacent to every vertex of $B$

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## Theorem (Erdős, Hajnal, Pach 2000)

For every graph $H$, there exists a constant $\delta(H) > 0$ s.t. for every $H$-free graph $G$ with $n$ vertices there exists two disjoint sets $A, B \subseteq V(H)$ with the following properties:

- $|A|, |B| \geq n^{\delta(H)}$, and
- either $A$ is complete to $B$, or $A$ is anticomplete to $B$.

# Bipartite variant - proof idea

**Lemma**

Let $G$ be a $k$-partite graph with vertex classes $V_1, ..., V_k$, s.t. $|V_i| = t^{k-1}$ for $t, k \geq 2$.
One of the following is holds:

1. There is $i \neq j$ s.t. $V_i$ and $V_j$ contain $t$-element subsets anticomplete to each other
2. $G$ contains a $k$-vertex clique

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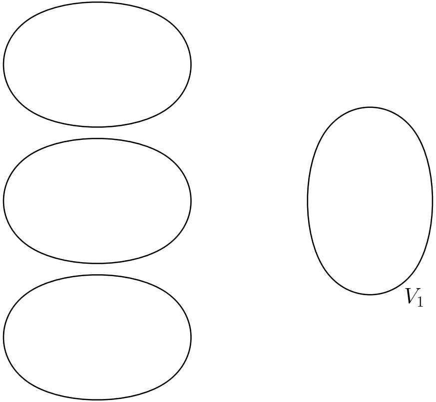
- Suppose that (1) doesn't hold.
- There exists a vertex $v_1 \in V_1$ with at least $t^{k-2}$ neighbours in each $V_i, i \geq 2$.
- We reduce the problem to $k-1$ classes with $t^{k-2}$ vertices each.

# Bipartite variant - proof idea

- Suppose, that for every $v \in V_1$, $v$ has at most $t^{k-2} - 1$ neighbours in $V_{i(v)}$ for $i(v) \neq 1$.



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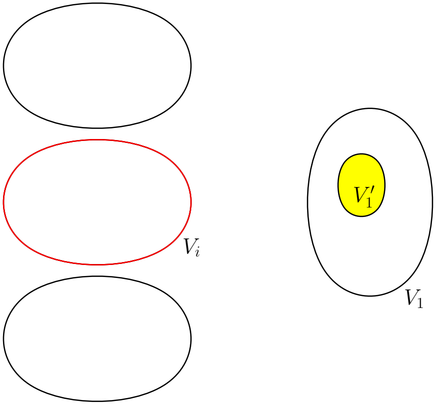
- Suppose, that for every $v \in V_1$, $v$ has at most $t^{k-2} - 1$ neighbours in $V_{i(v)}$ for $i(v) \neq 1$.
- Since $t^{k-1}/(k-1) \geq t$, we can find $i \neq 1$ and $t$-element subset $V_1' \subseteq V_1$, s.t. $i(v) = i$.



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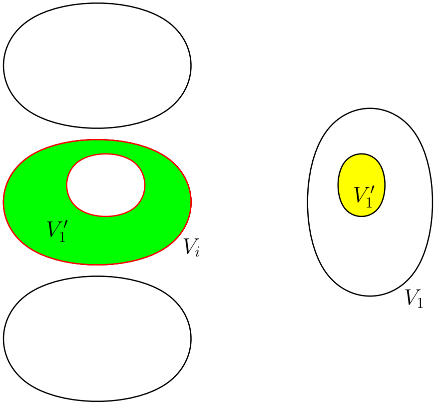
- Let $V_i'$ denote the set of vertices **not connected** to any element in $V_1'$.



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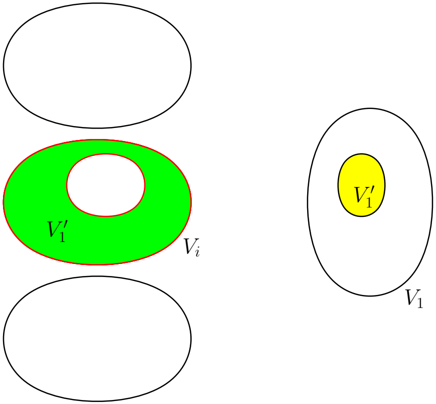
- Let $V_i'$ denote the set of vertices **not connected** to any element in $V_1'$.
- We have $|V_i'| \geq |V_i| - t(t^{k-2} - 1) = t$, and $V_1'$ is anticomplete to $V_i'$.



# Bipartite variant - stronger version

**Theorem (Fox, Sudakov 2010)**

For every graph $H$, there exists a constant $\delta(H) > 0$ s.t. for every $H$-free graph $G$ with $n$ vertices and $\omega(G) < n^{\delta(H)}$, there exists two disjoint sets $A, B \subseteq V(H)$ with the following properties:

- $|A|, |B| \geq n^{\delta(H)}$, and
- $A$ is anticomplete to $B$.

# Asymptotic result

**Theorem (Loebl et al. 2010)**

Let $\mathcal{F}_H^n$ be a class of all $H$-free graphs on $n$ vertices.
Let $\mathcal{Q}_H^{n,\epsilon}$ be a subclass of $\mathcal{F}_H^n$ consisting of graphs with $\omega(G) \geq n^\epsilon$ or $\alpha(G) \geq n^\epsilon$.
For every graph $H$, there exists a constant $\epsilon(H) > 0$ s.t. $\frac{|\mathcal{Q}_H^{n,\epsilon(H)}|}{|\mathcal{F}_H^n|} \to 1$ as $n \to \infty$.

# Tournament variant

## Definitions

- $\alpha(T)$ = the maximum size of an acyclic subtournament of $T$
- Tournament $T$ is $S-$*free* iff no subtournament of $T$ is isomorphic to $S$

## Conjecture equivalent to Erdős-Hajnal (Alon, Pach, Solymosi 2001)

For every tournament $S$, there exists a constant $\delta(S) > 0$ s.t. every $S$-free tournament $T$ with $n$ vertices satisfies $\alpha(T) \geq n^{\delta(S)}$.