(m, n)-cycle covers conjectures

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An (m, n)-cycle-cover of G is a multicoloring $c \colon E \mapsto \binom{[m]}{[n]}$ such that every subgraph induced by a color is a cycle.

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Throughout this presentation we assume that $n \leqslant m$.

Example

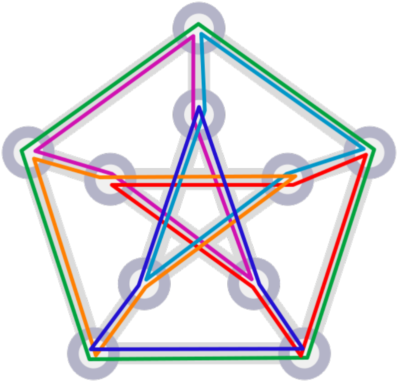

Figure: (6, 2)-cycle cover of Petersen graph.

Author: David Eppstein, picture is in public domain

Problem

Given positive integers n, m a natural question would be to ask what family of graphs have (m, n)-cycle-covers.

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For example, it is easy to answer this question whenever $n = m$, as such graphs must always be eulerian.

Known answers

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Odd n

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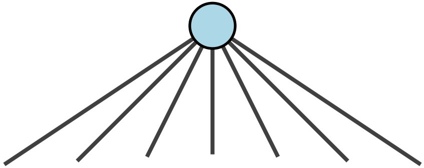

Bridges

The widest possible family of graphs that can have (m, n)-cycle cover are bridgeless graphs.

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A *bridge* (*cut-edge*) of graph is an edge such that deleting it would increase the number of connected components.

Are there positive integers m, n such that every bridgeless graph has (m, n)-cycle-cover?

Every bridgeless

Bermond, Jean-Claude

Graph is bridgeless if and only if it has $(7, 4)$-cycle-cover.

Bermond, Jean-Claude; Jackson, Bill; Jaeger, François *Shortest coverings of graphs with cycles.* J. Combin. Theory Ser. B 35 (1983), no. 3, 297–308.

Fan

Graph is bridgeless if and only if it has $(10, 6)$-cycle-cover.

Fan, Genghua *Integer flows and cycle covers.* J. Combin. Theory Ser. B 54 (1992), no. 1, 113–122.

Berge-Fulkerson

The Berge-Fulkerson conjecture

If G is a bridgeless cubic graph, then there exist 6 perfect matchings $M_1, \ldots, M_6$ of G with the property that every edge of G is contained in exactly two of $M_1, \ldots, M_6$.

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By Jaeger this conjecture is equivalent to:

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This is one better than the theorem on the previous slide (the same for $(7, 4)$-cycle-covers).

Bigger *n*s

Observation

If a graph G has both (m_1, n_1)-cycle-cover and (m_2, n_2)-cycle-cover then it has $(m_1 + m_2, n_1 + n_2)$-cycle-cover.

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Conclusion:

For each even $n \geqslant 4$ there exist m such that every bridgeless graph has (m, n)-cycle-covering.

Cycle double cover

Five cycle double cover conjecture

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Even weaker version of this theorem is still open

Cycle double cover conjecture

Graph is bridgeless if and only if it has $(\infty, 2)$-cycle-cover.

Nowhere zero flows

While all Eulerian graphs are the most restrictive cycle covers, and the bridgeless graphs the least there are also instances of (m, n)-cycle covers which are true for non-trivial families of graphs.

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First let us define a nowhere zero flow

Nowhere zero flow

A *flow* on a graph (V, E) is a function ϕ such that for each $v \in V$

$$\sum_{e \in \delta^+(v)} \phi(e) = \sum_{e \in \delta^-(v)} \phi(e),$$

where $\delta^+(v), \delta^-(v)$ denote the sets of edges respectively coming out of and into into v.
A *nowhere-zero flow* is flow such that for every $e \in E$ $\phi(e) \neq 0$.

We also need to define a k-flow

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A theorem by Tutte states

Tutte

Graph has a nowhere-zero 4-flow if and only if graph has a $(3, 2)$-cycle-cover if and only if graph has a $(4, 2)$-cycle-cover.

Tutte, W. T. *On the algebraic theory of graph colorings.* J. Combinatorial Theory 1 (1966), 15–50.

Results for small values of m, n

Odd values of n omitted.

		m									
		2	3	4	5	6	7	8	9	10	11
n	2	Eulerian	NZ 4-flow	NZ 4-flow	5CDC conj	open					
	4	-		Eulerian	5 post. sets	B-F conj	all				
	6	-				Eulerian	7 post. sets	?	?	all	

NZ k-flow = nowhere-zero k-flow - graph has flow that is nowhere zero and all edges have an integer flow between 0 and k

k post. sets = k postman sets - it is possible to partition the edges of G into k postman sets (Route inspection problem)

all = all eulerian graphs

Oriented cycle covers

Oriented cycle

An *oriented cycle* in directed graph (V, E) is a map $\phi : E \to \{-1, 0, 1\}$ such that for each $v \in V$, the sum of ϕ on the incoming edges is equal to the sum of ϕ on the outgoing edges.

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(m, n)-oriented-cycle-cover

For an even integer n, a (m, n)-oriented-cycle-cover of a graph G is a list of m oriented cycles so that every edge of G appears as a forward edge $n/2$ times and a backward edge $n/2$ times.

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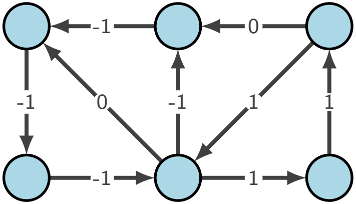

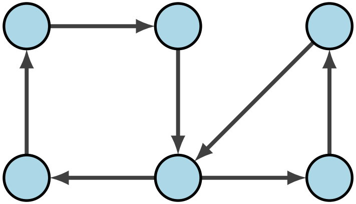

Orientable five cycle double cover

Similarly to unoriented case there exists conjecture about $(5, 2)$ case:

Orientable five cycle double cover conjecture

Graph is bridgeless if and only if it has $(5, 2)$-oriented-cycle-cover.

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And similarly even weaker conjecture is still open

Orientable five cycle double cover conjecture

Graph is bridgeless if and only if it has $(\infty, 2)$-oriented-cycle-cover.

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It is easy to see that if we take every cycle in (m, n)-cycle covers twice with opposite directions, then it is a valid $(2m, 2n)$-oriented-cycle covers.
Together with theorem about $(7, 4)$-cycle-cover for bridgeless graphs that gives us

Observation

Graph is bridgeless if and only if it has $(14, 8)$-oriented-cycle-cover.

$(11,6)$-oriented-cycle-cover

It has been shown by Matt DeVoss and Luis Goddyn that using previously proven Seymour's 6-flow theorem

NZ 6-flow

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For every even integer $n \geqslant 12$ there exists an m so that every bridgeless graph has an (m, n)-oriented-cycle-cover.

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Observation

For every even integer $n \geqslant 12$ there exists an m so that every bridgeless graph has an (m, n)-oriented-cycle-cover.

This question is still open for $n = 2, 4, 10$.

Orientable Conjectures

Conjectures for cases $n = 2, 4$:

O5CDC

Every bridgeless graph has a $(5, 2)$-oriented-cycle-cover.

O(8,4)

Every bridgeless graph has a $(8, 4)$-oriented-cycle-cover.

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