Double-critical graph conjecture

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Double critical graph

Definition

A connected simple graph G is called double-critical, if removing any pair of adjacent vertexes lowers the chromatic number by two.

Theorem

Every complete graph on n vertices is an n-chromatic double-critical graph.

Double critical graph conjecture

Conjecture (Erdös-Lovász Tihany)

For any graph G with $\chi(G) > \omega(G)$ and any two integers $a, b \geq 2$ with $a + b = \chi(G) + 1$, there is a partition (A, B) of the vertex set $V(G)$ such that $\chi(G[A]) \geq a$ and $\chi(G[B]) \geq b$.

Double-critical graph conjecture is a special case of the above conjecture.

Conjecture (Double critical graph conjecture)

Complete graph K_n is the only n-chromatic double-critical graph.

Properties

Definition

A graph G is called vertex-critical, if $\chi(G - v) < \chi(G)$ for every vertex $v \in V(G)$.

Theorem

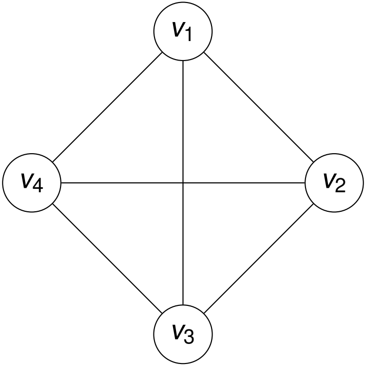
Every double-critical graph is also vertex-critical.

Lets take any edge $e = \{v, x\} \in V(G)$. G is double-critical so we can colour $G - \{v, x\}$ with $k - 2$ colours. Then can add x to the graph, colour x with $k - 1$, and we get a $k - 1$ colouring for $G - \{v\}$.

$K_{k \leq 4}$

Theorem

For $k \leq 4$ double-critical conjecture holds.



K_5

Theorem (Stiebitz 1987)

The only double-critical graph with chromatic number 5 is the complete graph on 5 vertices.

Definition

For graph G, and $x \in V(G)$ We denote $N(x : G)$ as the set of vertices of G, which are adjacent to x.

Definition

For edge $e = \{x, y\} \in E(G)$ we denote $T(e : G) = N(x : G) \cap N(y : G)$.

Proposition

For all edges $e = \{x, y\} \in E(G)$ and for all 3-colourings c of $G - \{x, y\}$, $|c(T(e : G))| = 3$. This implies, in particular, $|T(e : G)| \geq 3$ for all edges $e \in E(G)$.

Let $e = \{x, y\}$ be an edge of G, since G is double-critical $G - \{x, y\}$ is 3-colourable with some coloring c. We can extend c to 5-coloring h of G:

$$h(z) = \begin{cases} 4, & \text{if } z = x \\ 5, & \text{if } z = y \\ c(z), & \text{otherwise} \end{cases}$$

And now because G is 5-colourable there must exist some z_i that $h(N(z_i : G)) = \{1, 2, 3, 4, 5\} - \{i\}$.

Proposition

G contains a 4-clique.

Let $H_1, H_2, \ldots, H_r$ be a sequence of graphs of G such:

- H_i is a uniquely 3-colourable graph with i vertices.
- H_i is a subgraph of H_{i+1}.
- There is no uniquely 3-colourable subgraph of G with $r + 1$ vertices containing H_r.

The idea of the proof is that if we take $X = V(G) - V(H_r)$, and two vertices u, v of $T(e : G) - T(e : H_r)$, and $x, y \in H_r$ we know that both $H_r + u$ and $H_r + v$ and 4-chromatic, so $X - \{u\}$ and $X - \{v\}$ are independent sets of G, but X cannot be independent set of G, because G is $5-\mathit{chromatic}$, so $\{x, y, u, v\}$ is a 4-clique of G.

Double-critical graph conjecture

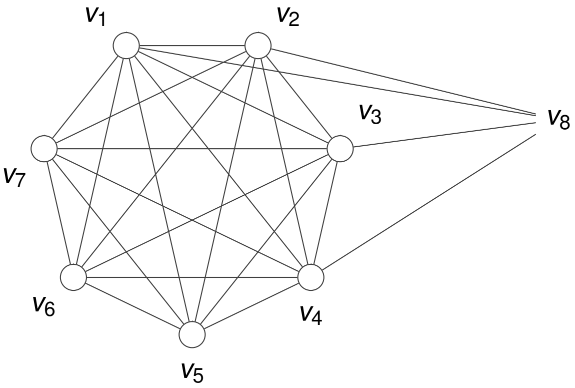
Open problem 1

Double-critical graph conjecture remains open for $k \geq 6$.

Non-complete double-critical graphs

Lemma

If G is k-chromatic non-complete double critical graph with $k \geq 6$, then G does not contain a complete $(k-1)$-graph as a subgraph.



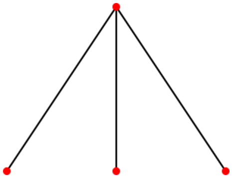
Results for 6-double critical graphs

Definition

A claw is a 4-vertex graph with one vertex of degree 3 and the others of degree 1. A graph is *claw – free* if it does not have a claw as an induced subgraph.

Theorem

Let G be a double-critical graph with $\chi(G) = 6$. If G is claw-free then $G \cong K_6$



Claw-free graphs

Lemma

Let G be a 6-double critical graph that is claw-free. If G is not a complete graph then for any $e = \{x, y\} \in E(G), 4 \leq |N(x : G) \cap N(y : G)| \leq 5$.

Lemma

Let G be a 6-double-critical graph that is claw-free. If $|N(x : G) \cap N(y : G)| = 4$ for some $\{x, y\} \in E(G)$, then $G \cong K_6$.

Lemma

Let G be 6-double-critical graph, and assume that G is claw-free. Suppose $|N(x : G) \cap N(y : G)| \geq 5$ for all $\{x, y\} \in E(G)$, then $G \cong K_6$.

Weaker conjecture

Conjecture

Every double-critical k-chromatic graph is contractible to the complete k-graph

Additional properties

Let G be a k-chromatic double-critical graph.

Theorem

For all edges $xy \in E(G)$ and (k-2) colourings of $G - x - y$ set of neighbours $N(x : G) \cap N(y : G)$ contains vertices of every colour class $i \in [k-2]$, and from that $|N(x : G) \cap N(y : G)| \geq k - 2$ and $\{x, y\}$ is contained in at least $k - 2$ triangles.

Proposition

For all vertices $x \in V(G)$, the minimum degree of the induced graph of neighbourhood of x in G is at least $k - 2$.

Non-complete double critical graphs

Proposition

Suppose G is a non-complete double-critical k-chromatic graph with $k \geq 6$. Then no minimal separating set S of G can be partitioned into two disjoint sets A and B such that $G[A]$ and $G[B]$ are edge-empty and complete respectively.

Theorem

Every double-critical k-chromatic non-complete graph is 6-connected.

Non-complete double critical graphs

Proposition

For any vertex $x \in V(G)$, there exists a vertex $y \in N(x)$ such that the set $N(x : G) \setminus N(y : G)$ is not empty.

Proposition

Every vertex of G has at least $k + 1$ neighbours.

Weaker conjecture - results

Theorem

Every double-critical 6-chromatic graph G contains K_6 as a minor.

Theorem

Every double-critical 7-chromatic graph G contains K_7 as a minor.

Open problem 2

Problem of proving the weaker double-critical conjecture for $k \geq 8$ remains open.

Similar problems

Definition (Double-edge critical graph)

A connected simple graph G is called double-edge critical, if removing any pair of non-incident edges lowers the chromatic number by two.

Definition (Mixed-double critical)

A connected simple graph G is called mixed-double critical, if removing any vertex $x \in V(G)$ and any edge in the graph $G - \{x\}$ lowers the chromatic number by two.

Theorem

A graph G is k-chromatic double-edge critical (mixed-double critical) if and only if it is the complete k-graph

Bibliography

1. K_5 ***is the only double-critical 5-chromatic graph***(1987) by Michael Stiebitz
2. ***A note on the double-critical graph conjecture*** (2016) by Hao Huang and Alexander Yu
3. ***Double-critical graphs and complete minors*** (2010) by Ken-ichi Kawarabayashi, Anders Sune Pedersen and Bjarne Toft