Hamiltonian paths/cycles in vertex-transitive/symmetric graphs

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17 grudnia 2020

Graph automorphism

- Automorphism of a graph $G = (V, E)$ is a permutation σ of the vertex set V, such that the pair of vertices (u, v) form an edge if and only if the pair $(\sigma(u), \sigma(v))$ also form and edge - graph isomorphism from G to itself.
- The composition of two automorphisms is another automorphism, and the set of automorphisms of a given graph, under the composition operation, forms a group, the automorphism group of the graph

Complexity of problems related to automorphism group

- Constructing the automorphism group is at least as difficult (in terms of its computational complexity) as solving the graph isomorphism problem.
- The *graph automorphism problem* is the problem of testing whether a graph has a nontrivial automorphism. Similar to the graph isomorphism problem, it is unknown whether it has a polynomial time algorithm or it is NP-complete.

Frucht's theorem

Theorem (Frucht, 1939 (conjectured by Konig in 1936))

Every finite group is the automorphism group of a finite undirected graph.

Theorem (Stronger version of Frucht's theorem)

For every finite group G there exist infinitely many non-isomorphic simple connected graphs whose automorphism groups are isomorphic to G.

Vertex-transitive graph

- Graph G is *vertex-transitive* if for any two vertices $v_1, v_2 \in V(G)$ there is some automorphism $\sigma : V(G) \to V(G)$ such that $\sigma(v_1) = v_2$.
- From the construction of σ, we see that $(v_1, u) \in E$ iff $(\sigma(v_1), \sigma(u)) = (v_2, \sigma(u)) \in E$.
- This gives more informal definition - a graph is vertex-transitive if every vertex has the same local environment, so that no vertex can be distinguished from any other based on the vertices and edges surrounding it.

Vertex-transitive graph

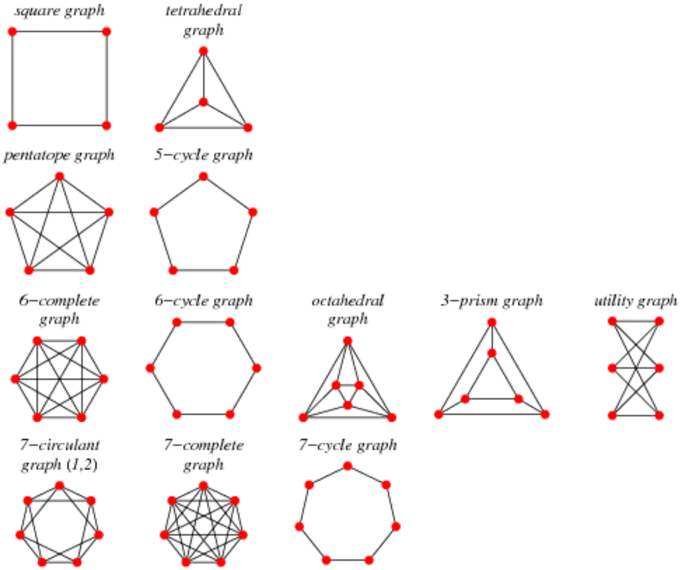
Fact

A graph G is vertex-transitive if and only if its graph complement G' is, since the automorphism groups of G and G' are identical.

Fact

Every vertex-transitive graph is regular - every vertice has the same number of neighbors.

Examples of vertex-transitive graphs



Lovász conjecture

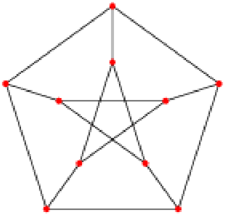
Conjecture (Lovász, 1969)

Every finite connected vertex-transitive graph contains a Hamiltonian path.

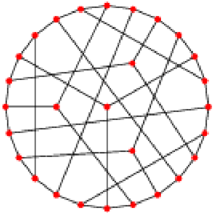
Examples of nonhamiltonian vertex-transitive graphs

2-path graph

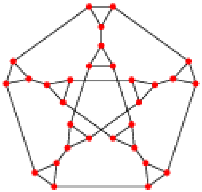
Petersen graph



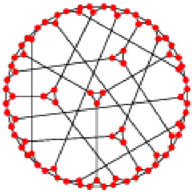
Coxeter graph



triangle-replaced Petersen graph



triangle-replaced Coxeter graph



Babai conjecture

Conjecture (Babai, 1996)

There exists $\epsilon > 0$ so that there are infinitely many connected vertex-transitive graphs G with longest cycle of length $< (1-\epsilon)|V(G)|$.

Theorem (Babai)

In every connected vertex-transitive graph exists cycle of length $\geq \sqrt{3|V(G)|}$.

Reminder from group theory

Generating set of a group

A subset S of a group G is said to be generating set for G if every element of the group can be expressed as a combination (under the group operation) of finitely many elements from S and their inverses.

Examples

- $(\mathbb{Z}_5^*, *)$ is generated by $\{2\}$
- $(\mathbb{Z}_m, +)$ is generated by $\{1\}$
- $(\mathbb{Z}, +)$ is generated by $\{p, q, -p, -q\}$ where $gcd(p, q) = 1$ (then we can find $x, y \in \mathbb{Z}$ such that $xp + yq = 1$)

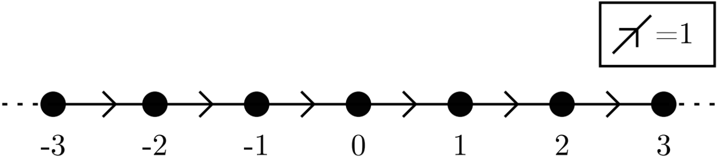
Cayley graph

Let G be a group and S - generating set of G. The Cayley graph $\Gamma = \Gamma(G, S)$ is colored directed graph constructed as follows:

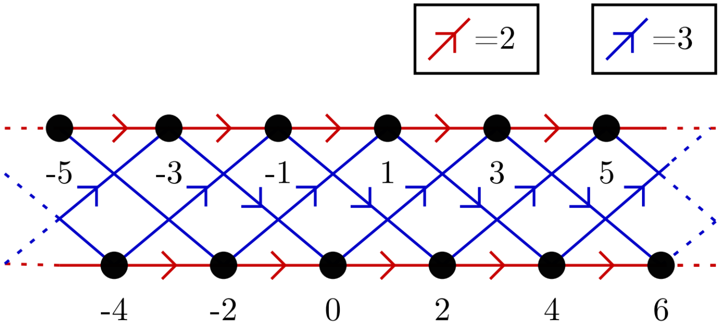
- Each element $g \in G$ is assigned vertex, $V(\Gamma) = G$.
- Each generator $s \in S$ is assigned a color c_s.
- For any $g \in G$, $s \in S$ there is directed edge (g, gs) of colour c_s.

Common assumption - S is finite, symmetric and does not contain the identity element of the group. In this case, the uncolored Cayley graph is an ordinary graph: its edges are not oriented and it does not contain loops.

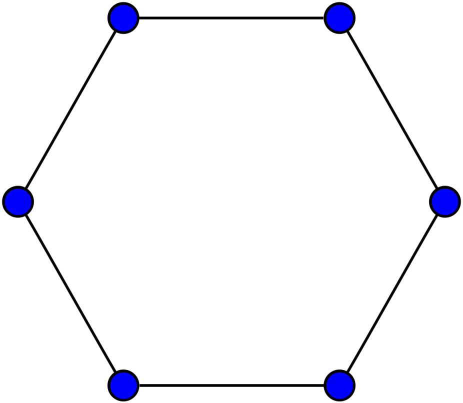
Cayley graph $\Gamma((\mathbb{Z},+),\{1,-1\})$



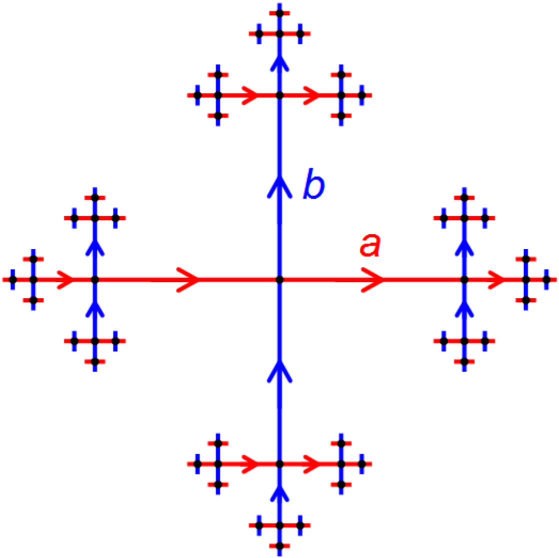
Cayley graph $\Gamma((\mathbb{Z}, +), \{2, 3, -2, -3\})$



Cayley graph $\Gamma((\mathbb{Z}_6, +), \{1, -1\})$



Cayley graph $\Gamma((F_{\{a,b,a^{-1},b^{-1}\}},\cdot),\{a,b,a^{-1},b^{-1}\})$



Variant of Lovász conjecture

Observation

None of the 5 vertex-transitive graphs with no Hamiltonian cycles is a Cayley graph.

Conjecture

Every finite connected Cayley graph with more than 2 vertices contains a Hamiltonian cycle.

Current results

For directed Cayley graphs the Lovász conjecture is false. More specifically, every cyclic group whose order is not a prime power has a directed Cayley graph that does not have a Hamiltonian cycle.

Theorem

On the other hand, every Cayley graph of a finite abelian group G such that $|G| \geq 3$ has a Hamiltonian cycle.

Babai conjecture

Theorem

Every finite group G of size $|G| \geq 3$ has a generating set S of size $|S| \leq log_2|G|$, such that the corresponding Cayley graph $\Gamma(G, S)$ contains a Hamiltonian cycle.

Observation

The bound on S is reached on the group $G = \mathbb{Z}_2^n$:

- $|G| = 2^n$
- *smallest generating set S has size n.*

Conjecture

Does every finite group have a minimum Cayley graph with a Hamilton cycle?

Random generating set

Conjecture

There exists $\epsilon > 0$, such that for every finite group G and every $k \geq \epsilon log_2|G|$, the probability $P(G,k)$ that the Cayley graph $\Gamma(G,S)$ with a random generating set S of size k contains a Hamiltonian cycle, satisfies $P(G,k) \to 1$ as $|G| \to \infty$.

Theorem (Krivelevich, Sudakov)

For every $\epsilon > 0$ a Cayley graph $\Gamma(T,S)$ with large enough $|G|$, formed by choosing a set S of $\epsilon log^5|G|$ random generators in a group G, is almost surely Hamiltonian.