From linear lambda terms to rooted trivalent maps

Mateusz Kaczmarek

Institute of Theoretical Computer Science
Jagiellonian University

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Lambda Terms

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Application

$(M\ N)$ is an application of N to M.

Lambda terms as trees

$\lambda a.(\lambda b.a(\lambda c.(c(\lambda d.b)b))(a(\lambda e.(\lambda f.\lambda g.e)(\lambda h.((\lambda i.eh)e)(\lambda j.\lambda k.kj)))))$

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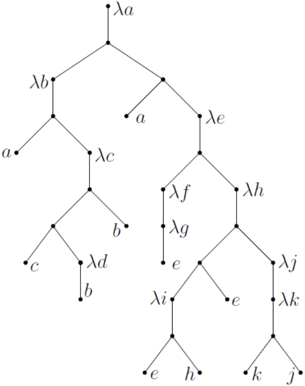

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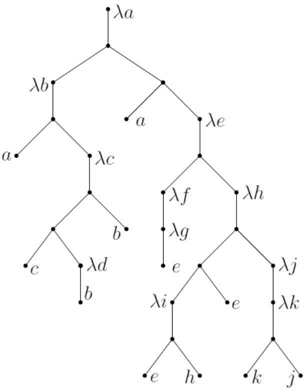
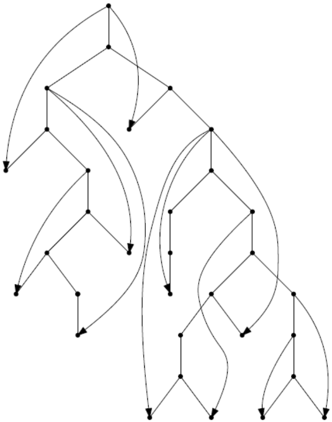
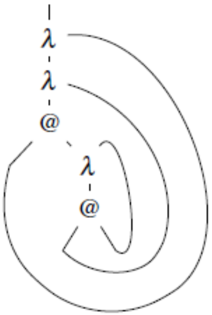




Lambda terms as trees

We can even go further and omit leaves.



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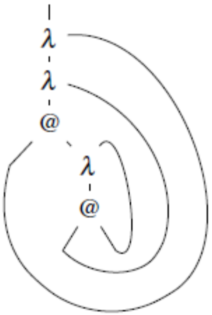

Which corresponds to:

$$\lambda x.(\lambda y(x(\lambda z(y\ z))))$$

Types of lambda terms

Closed lambda term

Term that contains no free variables.

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Linear lambda term

Bound variables occur once and only once i.e. each unary node is connected to exactly one leaf and vice versa.

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Planar lambda term

Is linear lambda term and its tree can be drawn without edge-crossing.

Maps

Definition

Map *is embedding* $i : G \to X$ *of an graph* G *into surface* X*: that is representation of vertices* $v \in G$ *by points* $i(v) \in X$ *and edges* v_1, v_2 *by arcs* $i(v_1), i(v_2)$ *such that no two arcs cross.*

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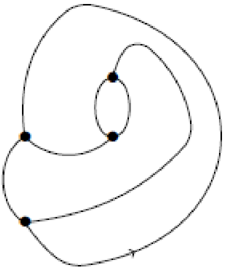
Rooted map is a map M with a distinguished element $r \in M$ called the **root**.

Example

Example of rooted trivalent map:



Some Results

Family of rooted maps	Family of lambda terms	OEIS entry
rooted trivalent maps	linear lambda terms	A062980
rooted planar maps	normal planar lambda terms	A000168
rooted maps	normal linear lambda terms / $\sim$	A000698

2013 - Bodini, Gardy, Jacquot
2015 - Zeilberger, Giorgetti
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Terms to Maps

We will try to show connection between linear lambda terms and rooted trivalent maps.

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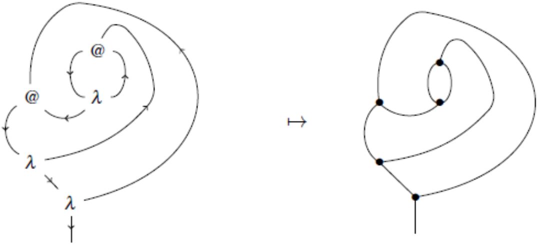
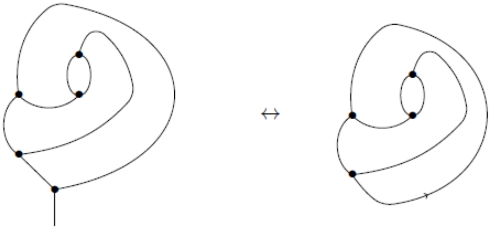
Forget distinction between @-nodes and λ-nodes, as well as the orientation of the edges:



$$\lambda x.(\lambda y(x(\lambda z(y\ z))))$$

Terms to Maps

We need to establish root:



Terms to Maps

What about free variables?

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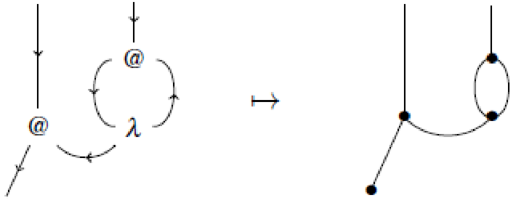

$$x(\lambda z(y\ z))$$

Terms to Maps

Theorem

To any linear lambda term with k free variables, p applications and q abstractions there is naturally associated a rooted trivalent map with boundary of degree k and $p + q$ trivalent vertices.

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Proof: Consider diagram of the term, which has k incoming edges (free variables), as well as p @-nodes and q λ-nodes. Transform @-nodes and λ-nodes into trivalent vertices, attach a univalent vertex to the end of the outgoing wire (wire out of root), and finally forget the orientations. The result is manifestly a rooted trivalent map with boundary of degree k and $p + q$ trivalent vertices. □

Map to Terms

Now, given map M, our task is to find a linear lambda term whose underlining rooted trivalent map is M.

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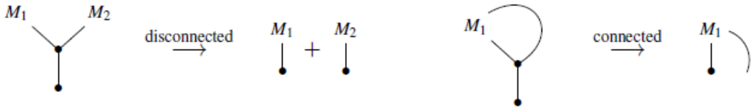
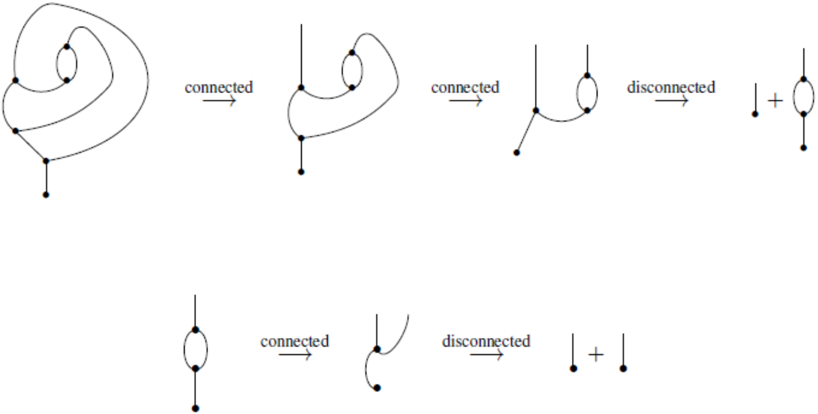
Observation

If removing trivalent vertex disconnects graph then the vertex corresponds to an @-node, and otherwise it corresponds to a λ-node.



Example


connected
connected
disconnected
+
connected
disconnected
+

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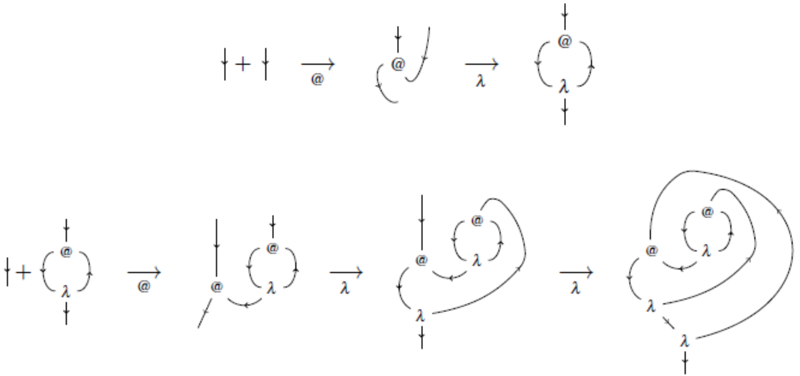

Maps to Terms

Theorem

To any trivalent map M with boundary of degree k and n trivalent vertices there is a unique linear lambda term with k free variables, p applications and q abstractions whose underlying trivalent map is M, for some $p + q = n$.

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Proof: As sketched before, and induction on n. □

Even more types of lambda terms

β-normal term

No beta reduction is possible i.e. left child is never a unary node.

3-indecomposable term

Tree stays connected after removing any two edges (ignoring root vertex).

Open problem

The map is bicubic iff. it is cubic (trivalent) and bipartite.

Conjecture

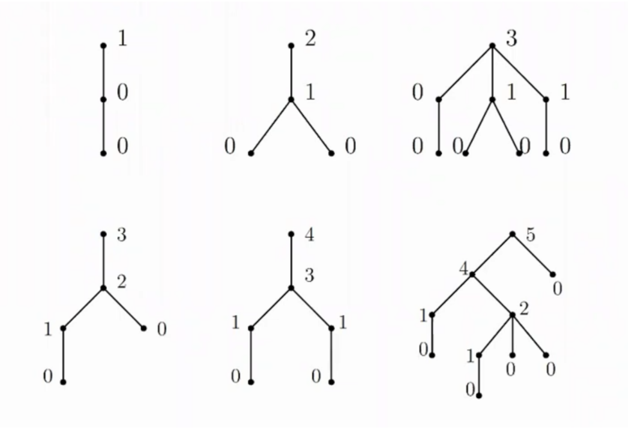
(Noam Zeilberger, 2019) The number of β-normal 3-indecomposable planar lambda terms is equal to the number of rooted bicubic map.

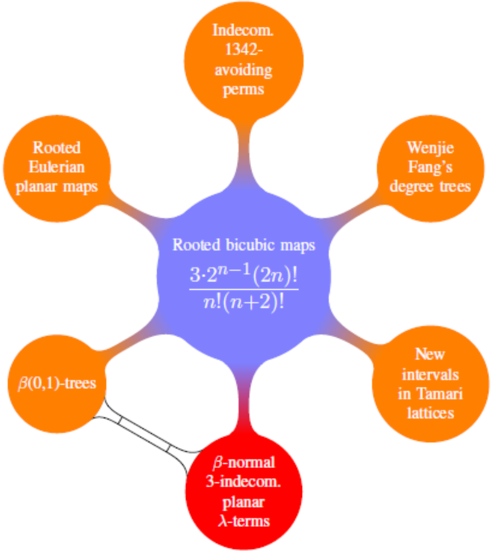
$\beta(0,1)$-trees

$\beta(0,1)$-trees are planar, rooted and labeled trees:

- leaves have label 0
- the label of the root is equal to the sum of its children's labels plus 1
- the label of any other node exceeds the sum of its children's labels by at most one.

Examples




Indecom. 1342-avoiding perms
Rooted Eulerian planar maps
Wenjie Fang's degree trees
Rooted bicubic maps
$\frac{3 \cdot 2^{n-1}(2n)!}{n!(n+2)!}$
β(0,1)-trees
New intervals in Tamari lattices
β-normal 3-indecom. planar λ-terms

Thank you!

Major number of ilustration were taken from:
Noam Zeilberger. Linear lambda terms as invariants of rooted trivalent maps. arXiv:1512.06751. 2015.