Bollobás-Eldridge-Catlin Conjecture

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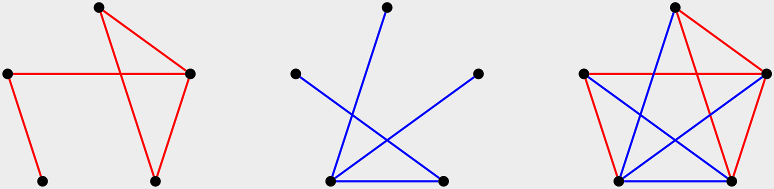

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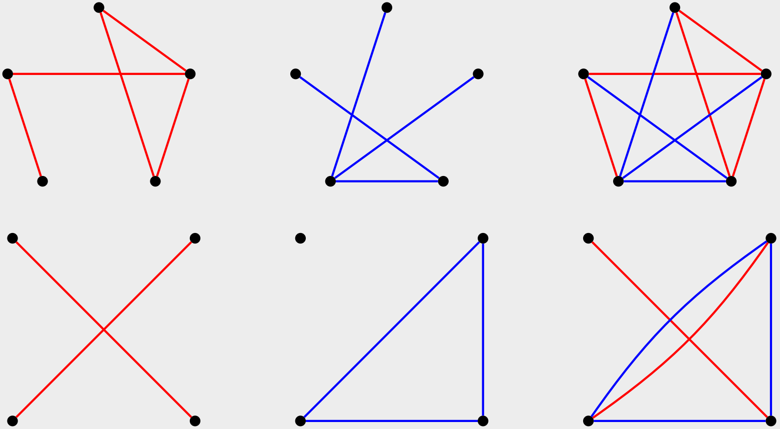

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- G contains a clique (resp. independent set) of size k if and only if $\overline{G}$ (resp. G) packs with $K_k \oplus I_{n-k}$,
- G is k-partite with part sizes $l_1, \ldots, l_k$ ($l_1 + \ldots + l_k = n$) if and only if G packs with $K_{l_1} \oplus \ldots \oplus K_{l_k}$.

Packing — example

Theorem (Ore)

If in n-vertex graph G for every pair of non-adjacent vertices u, v we have $\delta(u) + \delta(v) \geq n$, then G is Hamiltonian.

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In terms of packing:

If in n-vertex graph G for every pair of non-adjacent vertices u, v we have $\delta(u) + \delta(v) \leq n - 2$, then G packs with a cycle C_n.

Conjecture 1 (Bollobás, Eldridge (1978), Catlin (1974))

If

$$(\Delta(G_1)+1)(\Delta(G_2)+1) < n+1,$$

then G_1 *and* G_2 *pack.*

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Equivalently:

For any $k \leq n$ and any n-vertex graph G_1, if

$$\delta(\overline{G_1}) > \frac{kn-1}{k+1},$$

then $\overline{G_1}$ has as subgraphs all n-vertex graphs G_2 such that $\Delta(G_2) \leq k$.

Hajnal–Szemerédi Theorem

Theorem 1 (Hajnal, Szemerédi)

Every n-vertex graph G satisfying $\Delta(G) < k$ is k-colorable with every color occuring either $\lfloor \frac{n}{k} \rfloor$ or $\lceil \frac{n}{k} \rceil$ times.

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Proof.

For given k and G set H to be a sum of k disjoint cliques of size $\frac{n}{k}$. Since $(\Delta(G)+1)(\Delta(H)+1) \leq k \cdot \frac{n}{k} = n$, we get that G and H pack — but every ,,packed" clique in H corresponds to an independent set in G and we get desired result. $\square$

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Let's consider G_2's complementary $\overline{G_2}$. We have $\delta(\overline{G_2}) > \frac{n-1}{2}$ — determining whether G_1 and G_2 pack is equivalent to finding a matching of size $|E(G_1)| \leq \lfloor n/2 \rfloor$ in $\overline{G_2}$.

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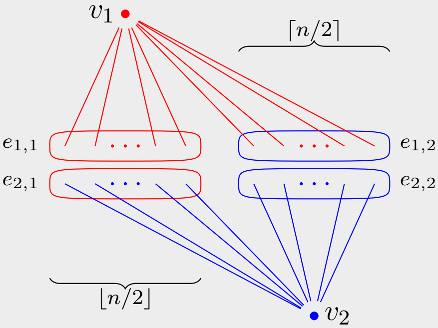
Let's arbitrarily split $\overline{G_2}$ into groups of $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ vertices. Then as long as the process increases the number of edges between groups, swap two vertices v_1, v_2 with minimal number of edges to the other group.

Case of $\Delta(G_1) \leq 1$ cont.



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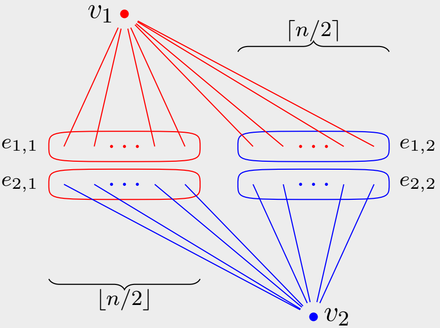

We have $e_{1,2} + e_{2,1} \geq e_{1,1} + e_{2,2}$, therefore
$e_{1,2} + e_{2,1} \geq \frac{1}{2}(e_{1,1} + e_{1,2} + e_{2,1} + e_{2,2}) > \frac{1}{2}(2 \cdot \frac{n-1}{2}) = \frac{n-1}{2}$.

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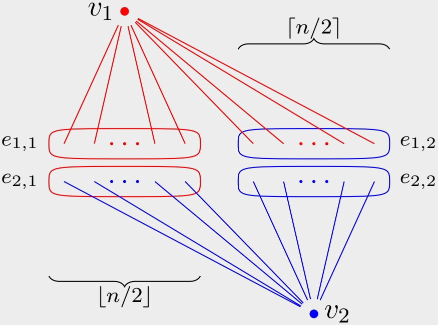

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Let R denote the set of red vertices. Now we apply Hall's theorem: let's take $A \subseteq R$. If $|A| \leq e_{1,2}$, then the condition is satisfied trivially. If $|A| > e_{1,2}$, then every blue v must have an edge leading to A (by pigeonhole principle: $\delta(v, R) + |A| > \frac{n-1}{2}$) and the condition holds as well, so we can match every red vertex.

Case of $\Delta(G_1) \leq 2$

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Theorem 2 (Catlin (1976))

If $\Delta(G_1) \leq 2$ *and* $\Delta(G_2) \leq \frac{n}{3} - \max(9, \frac{3}{2}n^{1/3})$ *then* G_1 *and* G_2 *pack.*

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Theorem 3 (Aigner, Brandt (1993))

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Case of $\Delta(G_1) \leq 3$

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Theorem 4 (Csaba, Shokoufandeh, Szemerédi (2003))

There exists an n_0 such that for all $n \geq n_0$ the following statement holds: Let H be a graph on n vertices with $\Delta(H) \leq 3$. If G is any n-vertex graph such that

$$\delta(G) \geq \frac{3n-1}{4},$$

then G contains H as a spanning subgraph.

Sauer-Spencer theorem

Theorem 5 (Sauer, Spencer (1978))

G_1 and G_2 pack if any of the following holds:

1. $|E(G_1)|, |E(G_2)| \leq n - 2$,
2. $|E(G_1)| \cdot |E(G_2)| < \binom{n}{2}$,
3. $\Delta(G_1) \cdot \Delta(G_2) < \frac{n}{2}$.

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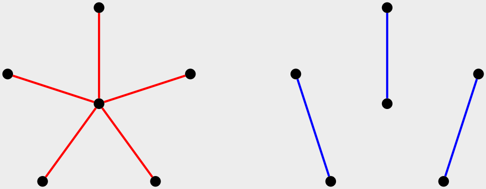
3. $\Delta(G_1) \cdot \Delta(G_2) < \frac{n}{2}$.

These conditions cannot be weakened for even n:

- For (1) and (2) we cannot pack a star ($n - 1$ edges) and a perfect matching ($n/2$ edges).



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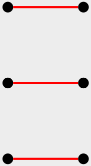

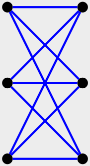

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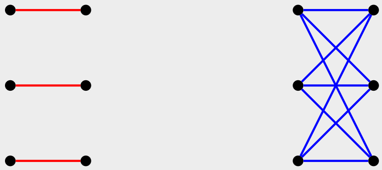
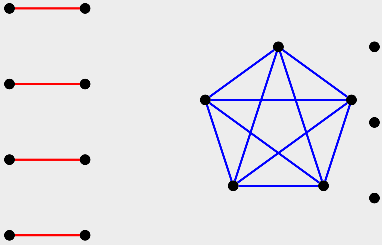


  - for even $\frac{n}{2}$: a $K_{n/2+1}$ and $\frac{n}{2} - 1$ isolated vertices.



Sauer-Spencer theorem cont.

Theorem 6 (Kaul, Kostochka (2007))

Let $\Delta(G_1) \cdot \Delta(G_2) \leq \frac{n}{2}$*. Then* G_1 *and* G_2 *do not pack if and only if one of them is a perfect matching and the other either:*

- *is a* $K_{n/2,n/2}$ *(with* $\frac{n}{2}$ *odd), or*
- *contains* $K_{n/2+1}$*.*

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Finally, another variation on the constant:

Theorem 7 (Kaul, Kostochka, Yu (2008))

Let $\Delta(G_1), \Delta(G_2) \geq 300$*. Then if* $(\Delta(G_1)+1)(\Delta(G_2)+1) \leq 0.6n+1$*, then* G_1 *and* G_2 *pack.*

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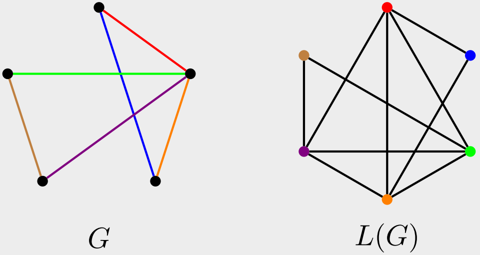

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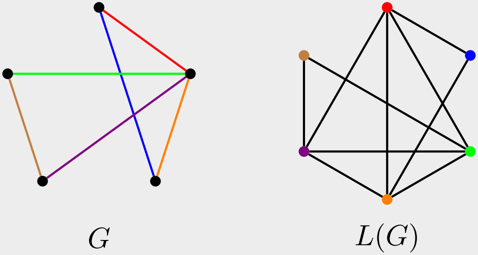

Let

$$\Theta(G) = \max\{\delta(u) + \delta(v) : uv \in E(G)\}.$$

Note that $\Theta(G) = \Delta(L(G)) + 2$.

Theorem 8 (Kostochka, Yu (2007))

If $\Theta(G_1)\Delta(G_2) \leq n$*, then* G_1 *and* G_2 *pack, unless either of the following is true:*

- G_1 *is a perfect matching and* G_2 *either is* $K_{n/2,n/2}$ *with* $\frac{n}{2}$ *odd or contains* $K_{n/2+1}$*,*
- G_2 *is a perfect matching, and* G_1 *either is* $K_{r,n-r}$ *with* r *odd or contains* $K_{n/2+1}$*.*

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