Polynomial Treedepth Bounds in Linear Colorings

Jedrzej Hodor

Jeremy Kun, Michael P. O'Brien, Marcin Pilipczuk, Blair D. Sullivan, 2020

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Basic definitions

For a graph G and its coloring ϕ, we define:

- Let H be a connected subgraph of G. If there exists $v \in H$ such that no other vertex $u \in H$ has a color $\phi(v)$, then we say that H **has a center** and, we call v the center.

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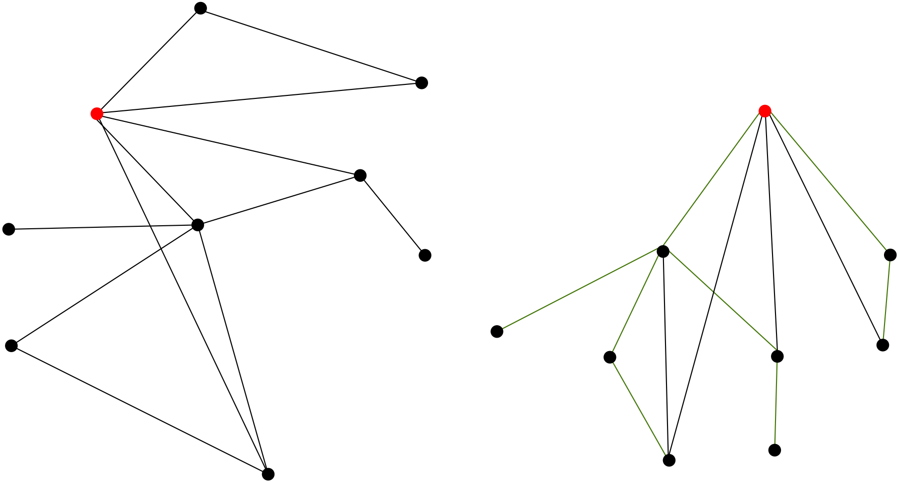
$$\mathrm{cen}(G) \leqslant 2 \cdot \mathrm{lin}(G)$$

Plan

1. Linear and centered colorings, treedepth and the problem statement.
2. When are they the same?
3. The lower bound.
4. Interval graphs and path width.
5. Trees.
6. General case.
7. Hardness of the LINEAR COLORING RECOGNITION problem.

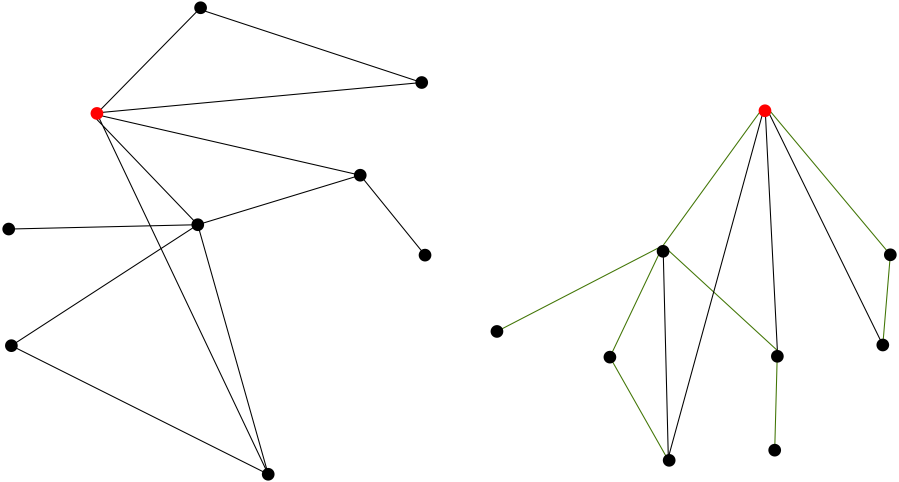
Treedepth

td(G) = minimal depth of an elimination tree



Treedepth vs Centered coloring

$$\mathrm{td}(G) = \mathrm{cen}(G)$$



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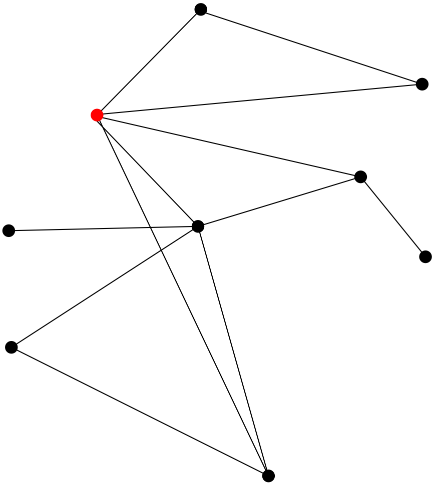

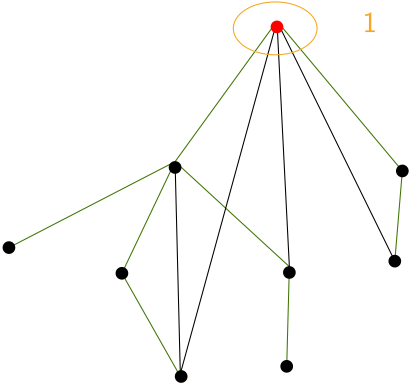

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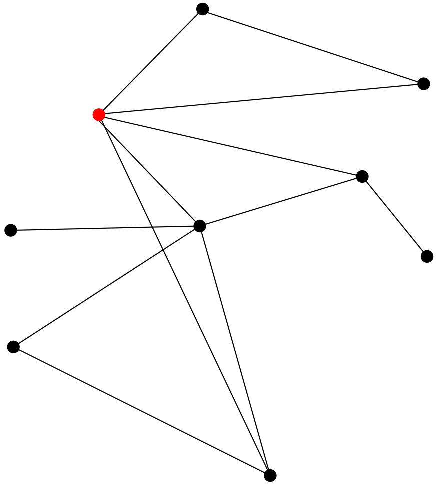

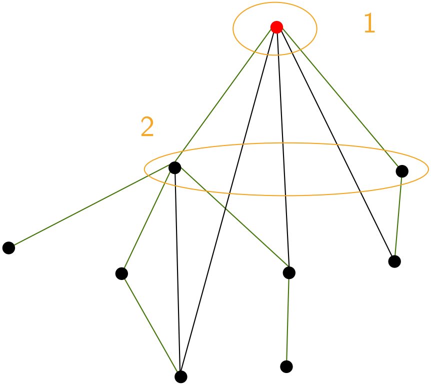

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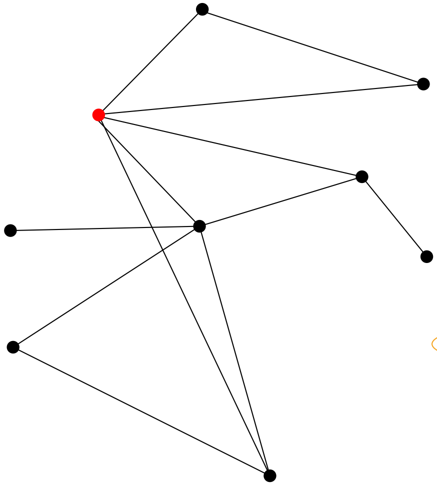

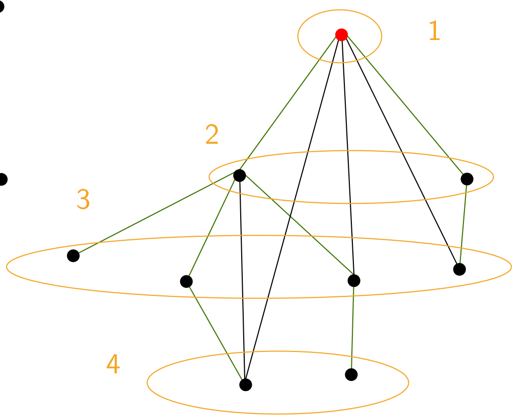

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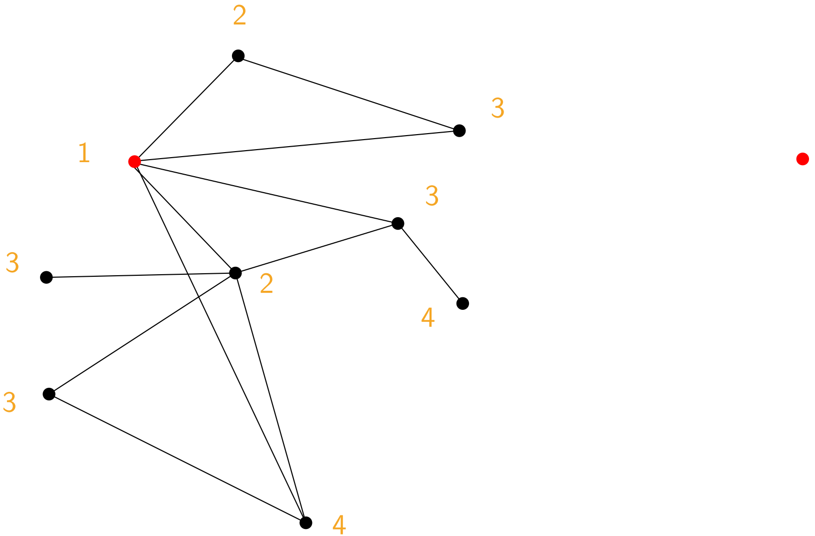

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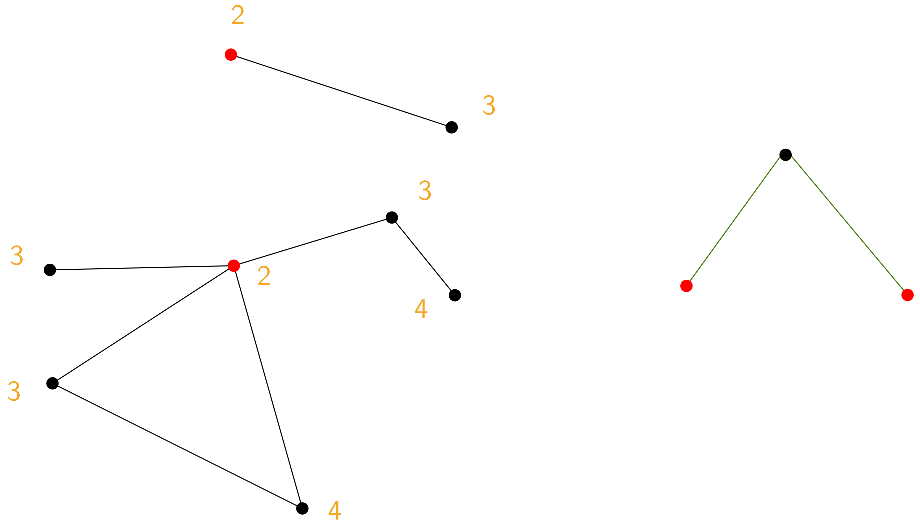
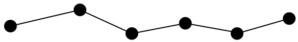
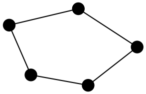


Centered = Linear

Paths, cycles





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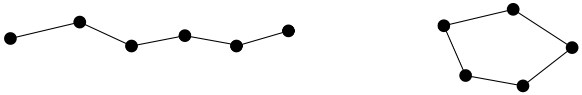

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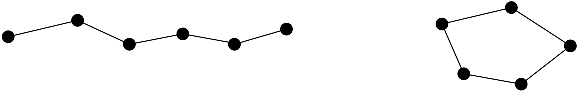

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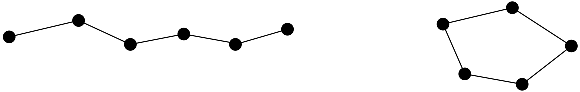
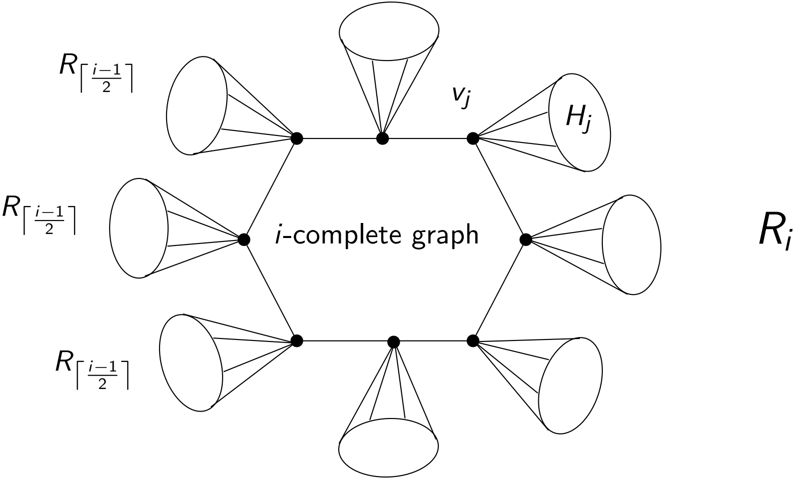


Cographs

Hereditary Hamiltonian path, for example: $\alpha(G) = 2$

Lower bound

- $\mathrm{lin}(R_i) = i$ (induction)
- $\mathrm{cen}(R_i) \sim 2i$ $(\mathrm{cen}(R_i) = i + \mathrm{cen}(R_p))$



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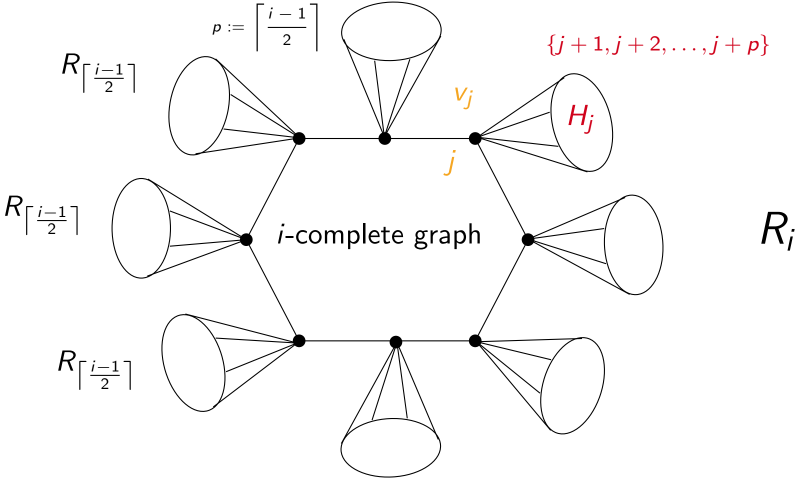

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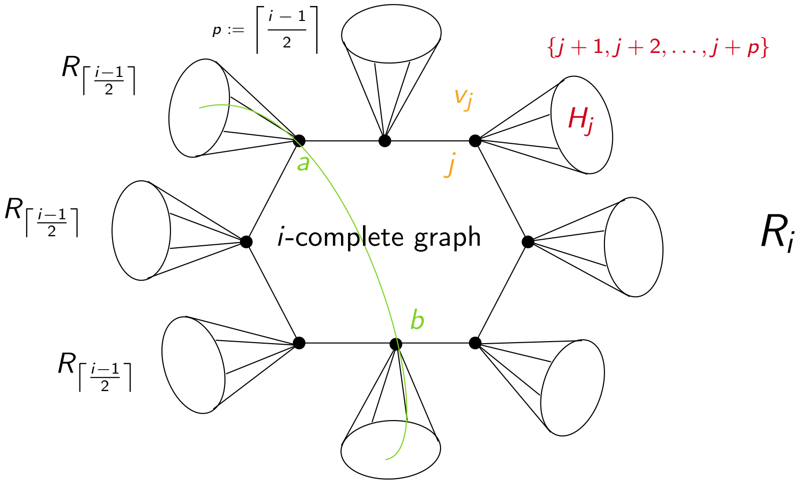

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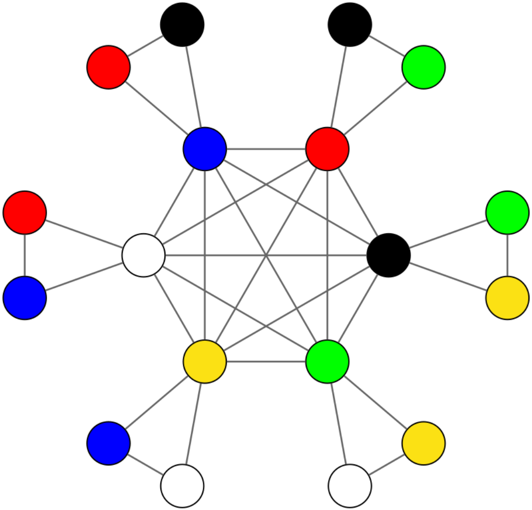
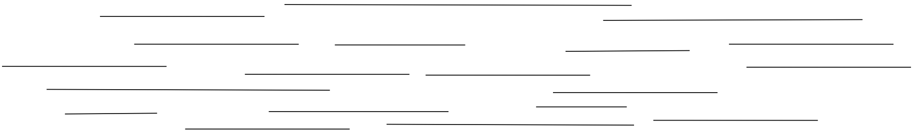


Interval graphs

- $\mathrm{cen}(G) \leqslant \mathrm{lin}^3(G)$



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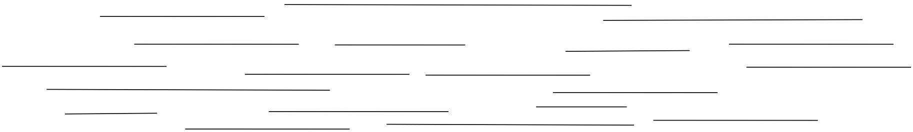

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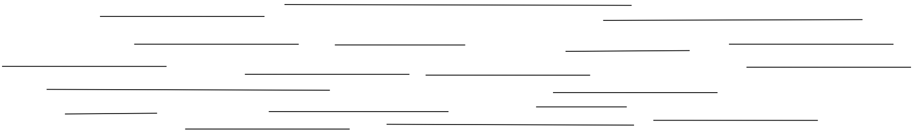

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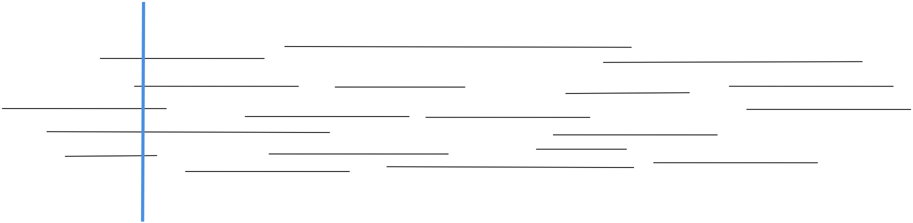

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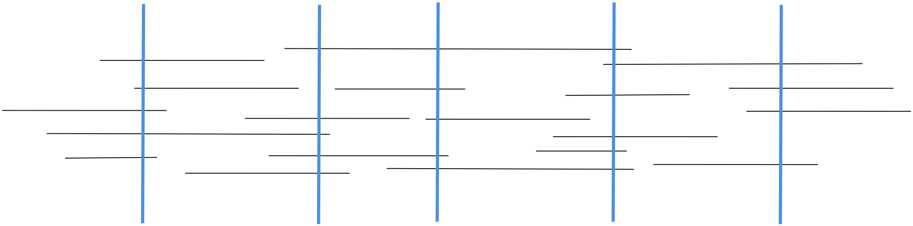

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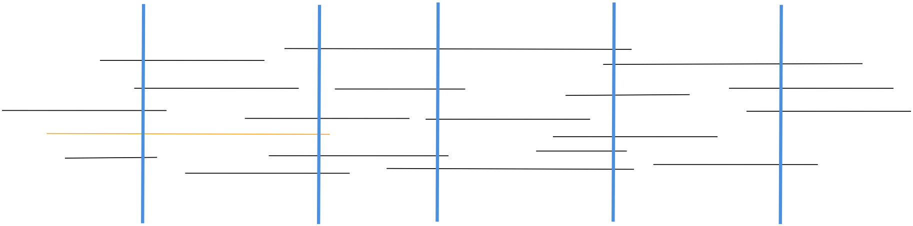

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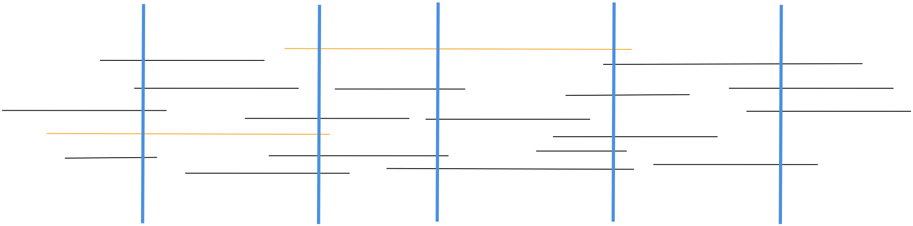

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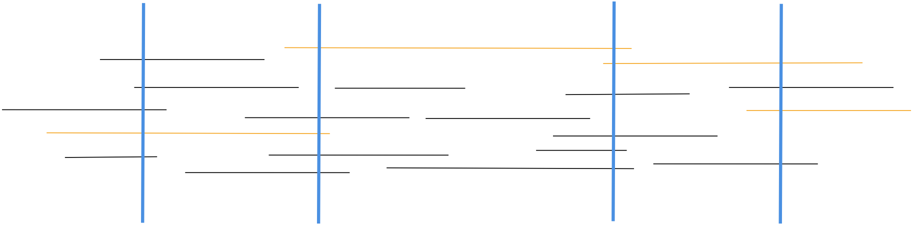

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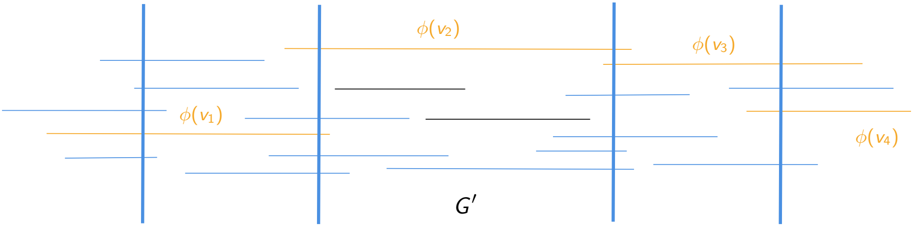

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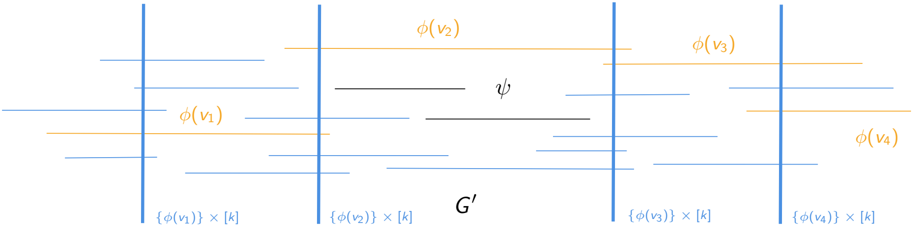

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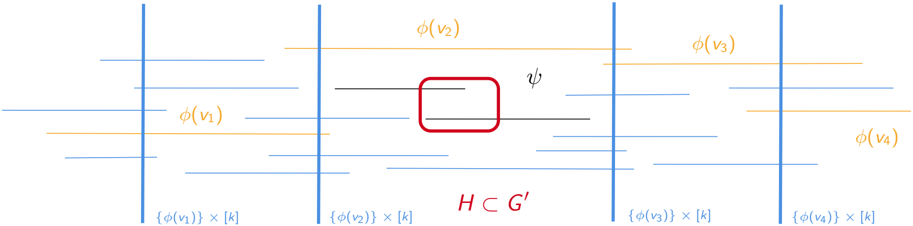

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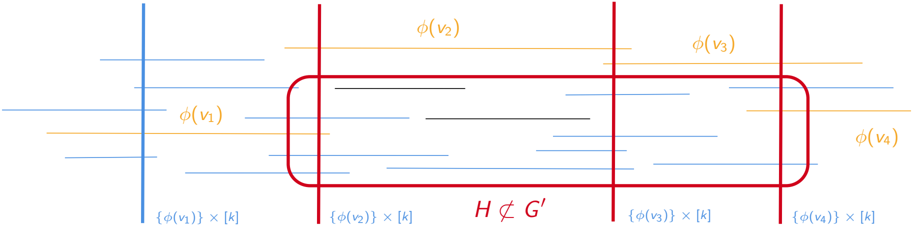

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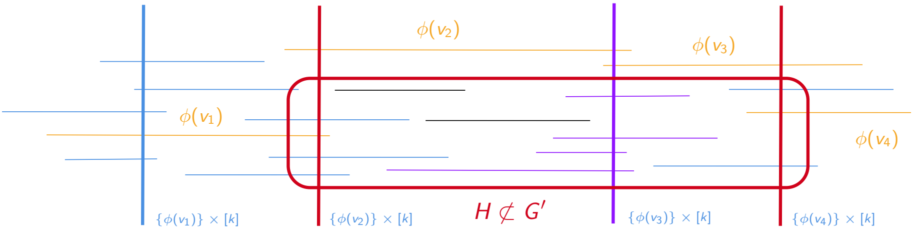

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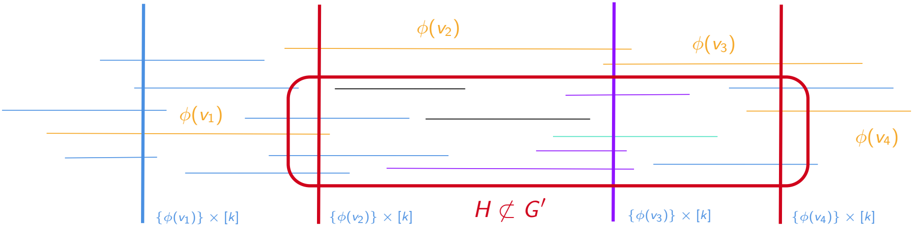

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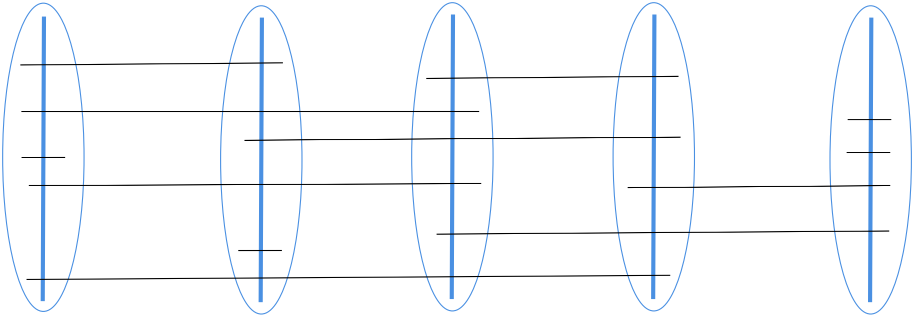

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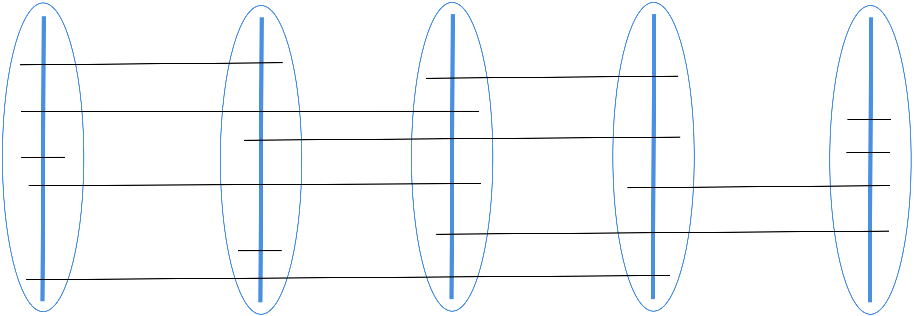

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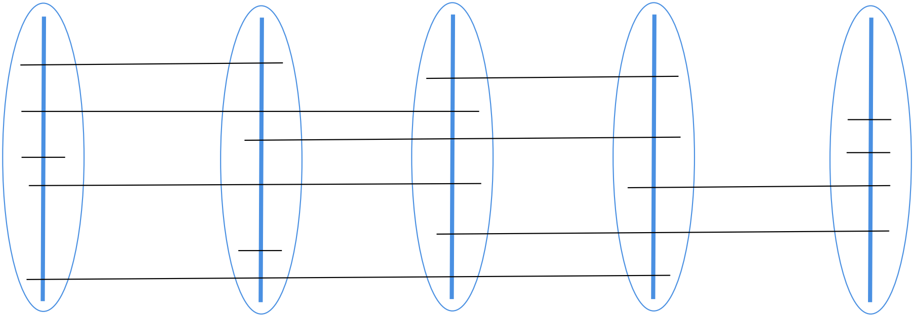

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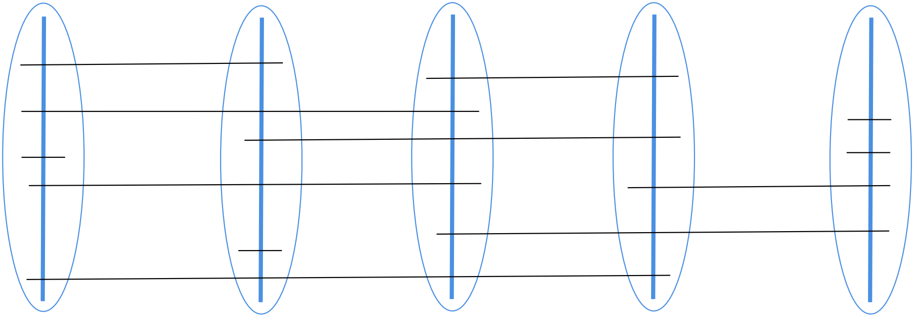

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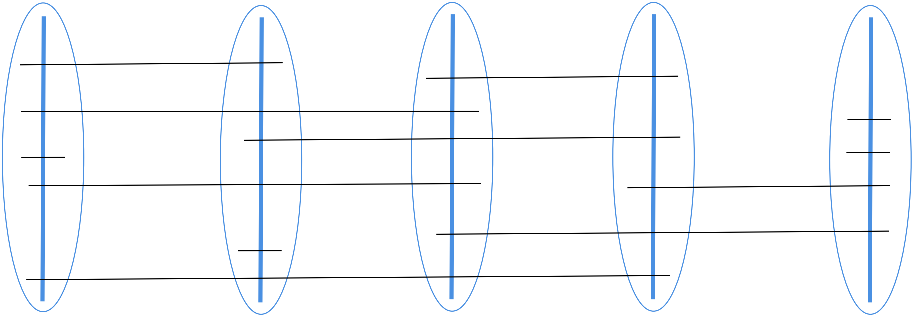

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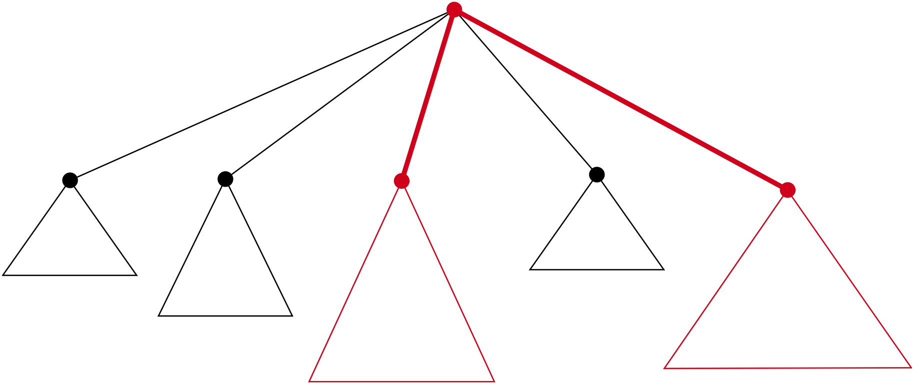

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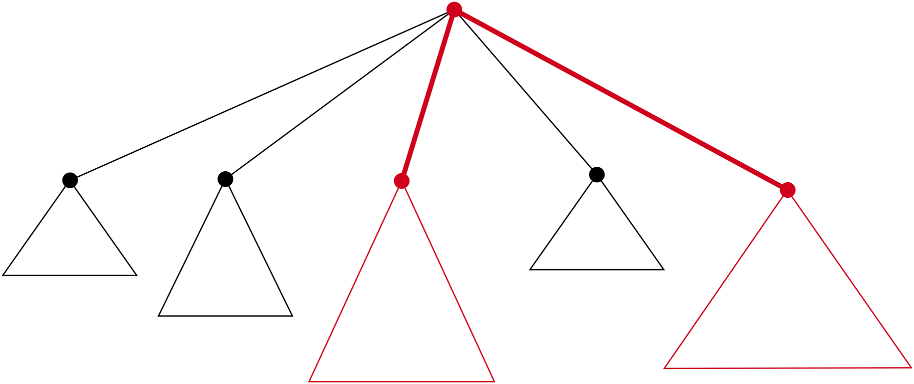
- $\mathrm{cen}(T) = \mathrm{td}(T) \leqslant \frac{\log 3}{\log \varphi}\mathrm{td}(C) \leqslant \frac{\log^2 3}{\log \varphi} \cdot \mathrm{lin}(C)$



The general case

Kawarabayashi, Rossman theorem (2018, upgraded by C,N,P)

If $\mathrm{td}(G) = C \cdot k^3$, then one of the following:

1. G contains 2^k - path
2. G contains k - binary tree (as a subdivision)
3. $\mathrm{tw}(G) \geqslant k$

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1. $k \leqslant \mathrm{lin}(G)$
2. $k \leqslant \mathrm{lin}(G) \cdot \log(3)$

The general case

Kawarabayashi, Rossman theorem (2018, upgraded by C,N,P)

If $\mathrm{td}(G) = C \cdot k^3$, then one of the following:

1. G contains 2^k - path
2. G contains k - binary tree (as a subdivision)
3. $\mathrm{tw}(G) \geqslant k$

We want: $\mathrm{cen}(G) \leqslant \mathrm{lin}^{54}(G)$

1. $k \leqslant \mathrm{lin}(G)$
2. $k \leqslant \mathrm{lin}(G) \cdot \log(3)$
3. More complicated...

Wrap up

The grid theorem (2019 - Chuzhoy, Tan)

If $\mathrm{tw}(G) \geqslant n^9$, then G contains $\boxplus_n$ as a minor.

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If G contains $\boxplus_n$ as a minor, then $\mathrm{lin}^2(G) \geqslant n$

Wrap up:

$\mathrm{td} = n^{54} \Rightarrow \mathrm{tw} \geqslant n^{18} \Rightarrow \boxplus_{n^2} \Rightarrow \mathrm{lin}^2 \geqslant n^2$

$$\mathrm{cen}(G) = \mathrm{td}(G) \leqslant \mathrm{lin}^{54}(G)$$

LINEAR COLORING RECOGNITION

- NON-CENTERED PATH - is there a path without center in G wrt ϕ?

LINEAR COLORING RECOGNITION

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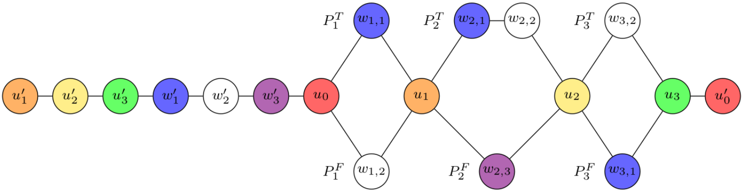
- Reduction from CNF-SAT $\Rightarrow$ *NP*-complete



LINEAR COLORING RECOGNITION

- NON-CENTERED PATH - is there a path without center in G wrt ϕ?
- Reduction from CNF-SAT $\Rightarrow$ *NP*-complete
- LINEAR RECOGNITION RECOGNITION is *coNP*-complete