Aanderaa-Karp-Rosenberg conjecture

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Query complexity and evasiveness of property

Throughout this presentation we will be considering undirected simple graphs unless otherwise specified.

Query complexity

Given a function on n bits $x_1, \ldots, x_n$ the query complexity of this function is the number of bits that need to be queried in order to determine the value of this function.

For example the or function has query complexity of n, because we could set all but one variable to 1 and thus algorithm that does not query it couldn't predict the outcome.
The title conjecture refers to functions on graphs that can query whether between two vertices an edge exists. Moreover we require that properties are invariant under graph-isomorphism.

Evasive property

A property of a graph is called *evasive* if its query complexity is $\frac{n(n-1)}{2}$.

Connectedness example

The aforementioned or function would be equivalent to non-emptiness property of graphs, since it checks whether an edge exists in graph. Are there other evasive properties?
Let's take for example connectedness. Is there an algorithm that can determine whether a graph is connected using $\frac{n(n-1)}{2} - 1$ queries or less? We may show that it is not a case by playing a game with an adversary [LBE74].

Is there edge between (a, c)? No.
Is there edge between (b, c)? No.
Is there edge between (c, d)? Yes.
Is there edge between (a, d)? No.
Is there edge between (a, b)? Yes.

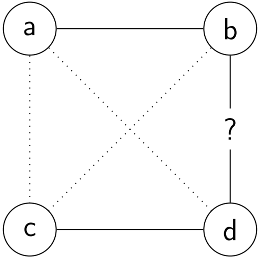

Image: [Com20b]

Connectedness strategy

Why will the adversary always win? We can consider two graphs: M and Y. M is the graph of edges that are confirmed to be in graph and those which were not yet queried. Y is the graph of edges that are confirmed to be in graph. Let us now notice the following facts:

1. Both Y and M are consistent with adversary answers.
2. Y is a subgraph of M.
3. M is connected.
4. If M has a cycle, none of its edges are in Y.
5. There are no cycles in Y.
6. If $Y \neq M$, then Y is disconnected.

First attempts at conjecture

The natural question to ask now is: which properties are evasive?

Non-trivial properties

A property of a graph is called *trivial* when either all graphs have it (e.g. "it has less than n^2 edges") or none have (e.g. "it has more than n^2 edges").

Are there any non-trivial properties which can be recognized in subquadratic time?

Rosenberg's conjecture [Ros73]

Any deterministic algorithm recognizing non-trivial graph property requires $\Omega(n^2)$ queries.

Scorpion graph

Let's consider a class of graphs called *scorpions*. A scorpion is a graph of size n consisting of a vertex b (the body) of degree $n-2$, vertex t (the tail) of degree 1 and a vertex u connected only to b and t. The remaining vertices may be in any configuration.

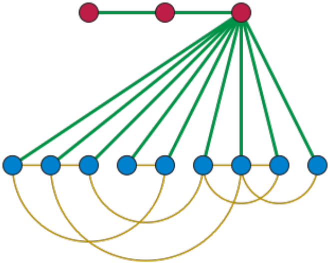

Figure: An example of a scorpion graph [Com20a].

Observation [LBE74]: it is possible to check whether graph is a scorpion using at most $6n$ queries.

Conjecture refinement

Rosenberg was already aware that the original conjecture does not work due to counterexamples provided by Aanderaa. Hence a refinement was proposed.

Monotone property

A property $\mathcal{X}$ of graph is called *monotone* if for each graph (V, E) in $\mathcal{X}$, graph V, E' such that $E \subseteq E'$ is also in $\mathcal{X}$.

The refined conjecture states:

Aanderaa-Rosenberg conjecture [Ros73]

Any deterministic algorithm recognizing non-trivial monotone graph property requires $\Omega(n^2)$ queries.

Results

This conjecture turned out to be true.

First proof by Rivest and Vuillemin [RV75]

Every non-trivial monotone property requires at least $\frac{n^2}{16}$ queries.

Later results:

- Kleitman and Kwiatkowski showed lower bound of $\frac{n^2}{9}$ [KK80]
- Kahn, Saks and Sturtevant showed $\frac{n^2}{4} - o(n^2)$ [KSS83]
- Korneffel and Triesch showed $\frac{8}{25}n^2 - o(n^2)$ [KT10]
- Latest result is by Scheidweiler and Triesch: $\frac{n^2}{3} - o(n^2)$ [ST13]

Back to evasiveness

At the same time the previous conjecture was introduced, Richard Karp already conjectured the stronger version of it:

Aanderaa-Rosenberg-Karp conjecture [Ros73]

Any deterministic algorithm recognizing non-trivial monotone graph property is evasive (requires $\frac{n(n-1)}{2}$ queries).

This conjecture is open to this day.

Evasiveness results

For several special graph classes it has been shown that all such properties are evasive:

- The evasiveness conjecture was proved for graphs with size that is a power of prime numbers [KSS83].
- Another class of graphs for which the conjecture was proven is bipartite graphs [Yao88].

Another strategy was to strengthen the properties assumptions.

- Properties that are always false when the graph contains either a 3-cycle or a 4-cycle [Tri94]
- Properties that are always false when the graph is not bipartite [Tri96]
- Properties that are closed under taking minors [CKS01]

What about randomized algorithms?

We can also approach the problem by dropping the requirement for deterministic algorithm.

Randomized algorithm query complexity

A randomized algorithm is said to have query complexity g if the expected number of queries before it provides correct answer is $g(n)$.

A general bound for any querying complexity is known

Relationship between deterministic and randomized complexity [BI87]

For any n-bit boolean function f, deterministic algorithm has complexity $g(n)$ if and only if there exists randomized algorithm that has query complexity $\leqslant g(n)^2$.

Thus we can conclude that any randomized algorithm will have at least complexity $\sqrt{\frac{n^2}{16}} = \Omega(n)$.

Randomized continued

But can we do better for graphs? It was conjectured by Karp that for monotone nontrivial properties lower bound is still $\Omega(n^2)$ [SW86]. Currently all studied nontrivial monotone properties require at least $\frac{n^2}{4}$ queries. [Die92]
The first lower bound that was better than linear was proved by Yao [Yao91]. It was shown to be $\Omega(n \log^{\frac{1}{12}} n)$. This was later improved:

- By King [Kin88] to $\Omega(n^{\frac{5}{4}})$
- By Hajnal [Haj91] to $\Omega(n^{\frac{4}{3}})$
- And to current best known result $\Omega(n^{\frac{4}{3}} \log^{\frac{1}{3}} n)$, by Chakrabarti and Khot [CKS01].

Critical probability

Some newer results rely on the notion of critical probability to improve lower bounds.

Critical probability

Given a property of a graph, a *critical probability* of it is p such that a random graph $G(n, p)$ (graph where each edge is chosen with probability p) has this property with probability $\frac{1}{2}$.

Using this notion it was shown that

Kahn and Wigderson [FKW02]

Any monotone property with critical probability p requires

$$\Omega\left(\min\left\{\frac{n}{\min(p, 1-p)}, \frac{n^2}{\log n}\right\}\right)$$

queries

Special properties

Similarly to the deterministic case we may try to restrict the conjecture only to special cases.

Colourability [Die92]

For any k checking whether a graph is k-colourable requires $\Omega(n^2)$ queries.

The same paper also shows a weaker lower bound for a more general family of properties

Subgraph isomorphism properties [Die92]

Randomized algorithm for checking any subgraph isomorphism property will require $\Omega(n^{\frac{3}{2}})$ queries in the worst case.

Quantm query complexity

The problem was also studied in the realm of quantum computation. But while the deterministic and randomized algorithms had conjectured lower bounds of $\Omega(n^2))$ we know that this lower bound does not hold for quantum algorithms.

Grover's algorithm [Gro96]

Given a black box function f and the output O of the function, the algorithm finds an input I for the function such that $f(I) = O$ in time $\mathrm{O}(\sqrt{|D_f|})$, where D_f is the domain of f with high probability.

This algorithm can be used to test the non-emptiness property using $\mathrm{O}(n)$ queries. Thus the conjectured lower bound for such properties are linear.
The best current result providing lower bound for deciding monotone properties is $\Omega(n^{\frac{2}{3}} \log^{\frac{1}{6}} n)$ by Yao [MSS03].

Specific properties

Still there exist properties that have much higher lower bound than linear. For example it was shown that

- Nonplanarity property requires $\Theta(n^{\frac{3}{2}})$ queries on quantum computer [Amb+08].
- Property recognizing graphs that contain more than $\frac{n(n-1)}{4}$ edges requires $\Theta(n^2)$ queries on quantum computer [Bea+01].

Fin

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