# An Introduction to the Discharging Method via Graph Coloring

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May 13th, 2021

# Presentation overview

- Introduction – what is the discharging method about
  - Example: a class of graphs in which $\chi'(G) = \Delta(G)$
- Structure and coloring of sparse graphs
  - Remark: optimal value for discharging arguments
  - Acyclic 6-choosability
  - Star colorability and Timmons' theorem
- Discharging on plane graphs
  - Acyclic 6-colorability for plane graphs
  - Four color theorem
- List Coloring
- Edge-coloring and List Edge-coloring

# Introduction

In the simplest version of discharging involves just reallocation of vertex degrees in the context of a global bound on the average degree. We view each vertex as having an initial "charge" equal to its degree. To show that average degree less than $b$ forces the presence of a desired local structure, we show that the absence of such a structure allows charge to be moved (via "discharging rules") so that the final charge at each vertex is at least $b$. This violates the hypothesis, and hence the desired structure must occur.

# Definitions

## Configuration

A **configuration** in a graph $G$ can be any structure in $G$ (often in specified sort of subgraph). A configuration is **reducible** for a graph property $Q$ if it cannot occur in a minimal graph not having property $Q$. Let $d(v)$ denote the degree of $v$ in $G$, and $\overline{d}(G)$ denote the average of the vertex degrees in $G$.

**Degree charging** is the assignment to each vertex $v$ of an "initial charge" equal to $d(v)$

## Example (a silly one)

$K_3$ is a reducible configuration for graph property $\chi(G) = 2$

To put it simple, a word *reducible* means that if $G$ does not have a certain property $Q$, we can remove a configuration from $G$ and the obtained graph will still not have $Q$.

# Definitions cd.

## Vertices

A *j-vertex, $j^+$-vertex or $j^-$-vertex* is a vertex with degree equal to $j$, at least $j$ or at most $j$, respectively. A $j$-neighbor of $v$ is a $j$-vertex that is a neighbor of $v$.

## Weight of an edge

The **weight** of a subgraph $H$ of a graph $G$ is $\sum_{v \in V(H)} d_G(v)$; we sum the degrees in the full graph G

## Maximum average degree

$\text{mad}(G) = \max_{H \subset G} \overline{d}(H)$. Note that $\overline{d}(G) \leq \text{mad}(G)$

# Example

## Lemma 1.3

If $\overline{d}(G) < 3$ then $G$ has a $1^{-}$-vertex or a 2-vertex with a $5^{-}$-neighbor

## Lemma 1.4

An edge with weight as most $k+1$ is a reducible configuration for the property of being $k$-edge-colorable. That is, if $G$ has edge of weight at most $k+1$ then:

- $G$ is $k$-edge-colorable, OR
- $G$ is not $k$-edge-colorable, but we can remove an edge $e$ and the resulting graph $G' \subset G$ will also **not be** $k$-edge-colorable

## Theorem 1.6

If $\mathrm{mad}(G) < 3$ and $\Delta(G) \geq 6$, the $\chi'(G) = \Delta(G)$. We will prove that for $k \geq 6$ if $\mathrm{mad}(G) < 3$ and $\Delta(G) \leq k$ then $\chi'(G) \leq k$.

# Sparse graphs

## Remark – best bound on mad($G$)

Let's try to formulate lemma 1 more generally:
If $\overline{d}(G) < b$ then $G$ has a $1^-$-vertex or a 2-vertex with a $j^-$-neighbor. We must make $b \leq 3$, othwerise a 3-regular graph would be a counterexample Let's say each 2-vertex takes $\rho$ from each neighbor. We want final charge to be equal at least $b$ to have a proof by contradiction. We can have it iff. 2-vertices obtain enough charge, and vertices with degree larger than $j$ do not lose too much. So we need $2 + 2\rho \geq b$ and $d - d\rho \geq b$ when $d \geq j + 1$. To find largest $b$ that works, set $2 + 2\rho = (j + 1)(1 - \rho)$ yielding $b = 2 + 2\rho = 2\frac{j+1}{j+3}$, which gives lemma 1 for $j = 5$

# Acyclic colorings

**Definition**

An *acyclic coloring* of a graph is a proper coloring such that the union of any two color classes induces an acyclic subgraph; equivalently, no cycle is 2-colored.

**Theorem 2**

If $\mathrm{mad}(G) < 3$ then $G$ is acyclically 6-choosable

# Note about structure

## Lemma 1(aka. "structure theorem")

If $\overline{d}(G) < 3$ then $G$ has a $1^{-}$-vertex or a 2-vertex with a $5^{-}$-neighbor

## What if $\overline{d}(G)$ exceeds 3?

If a structure theorem with hypothesis $\overline{d}(G)$ is sharp, then when $\overline{d}(G)$ exceeds 3, we must add other configurations to obtain a structure theorem

## What we strenghten the bound on $\overline{d}(G)$?

We can impose more sparseness. For example, if $\overline{d}(G) < \frac{12}{5}$ then $G$ has two adjacent 2-vertices if it has no $1^{-}$-vertex.

# $\ell$-Threads

## An $\ell$-thread definition

An $\ell$-thread in a graph $G$ is a trail of length $\ell + 1$ in $G$ whose $\ell$ internal vertices have degree 2 in the full graph $G$.

## Lemma 2.5

*If $\overline{d}(G) < 2 + \frac{2}{3\ell - 1}$ and $G$ has no 2-regular component, then $G$ contains a $1^-$-vertex or an $\ell$-thread*

## Proof

Let $p = \frac{1}{3\ell - 1}$ so the hypothesis is $\overline{d}(G) < 2 + 2p$. Let's suppose that neither configuration occurs. Redistribute charge to leave each vertex with at least $2 + 2p$. Since $G$ has no $1^-$-vertex, $\delta(G) \geq 2$. Since $G$ has no cycle, each 2-vertex lies in a unique maximal thread.

# Star colorability

A *star coloring* is an acyclic coloring where the union of any two color classes induces a forest of stars; equivalently, no 4-vertex path is 2-colored. The *star chromatic number* $s(G)$ (also written $\chi_s(G)$) is the minimum number of colors in such a coloring.

- Every star coloring is an acyclic coloring
- All trees are acyclically 2-colorable
- Trees of diameter at least 3 are not star 2-colorable

# Star colorability cd.

## $I, F$-partition definition

A set $I$ of vertices is a 2-independent set if the distance between any two vertices of $I$ exceeds 2. An $I, F$-partition of a graph $G$ is a partition of $V(G)$ into sets $I$ and $F$ such that $I$ is a 2-independent set and $G[F]$ is a forest.

## Lemma 2.18

Every forest is star 3-colorable. Hence, if a graph $G$ has an $I, F$-partition, then $s(G) \leq 4$

# Timmons' theorem

**Lemma 2.18**

Every forest is star 3-colorable. Hence, if a graph $G$ has an $I, F$-partition, then $s(G) \leq 4$

**Timmons' thm (2008)**

*If $mad(G) < \frac{7}{3}$ then $G$ has an $I, F$-partition*

*Proof.*

- We can assume that no component is a cycle
- Without cycles, lemma B with $t = 2$ implies that $G$ has a:
  - $1^-$-vertex
  - a 3-thread or
  - a 3-vertex with at least five 2-vertices on its incident threads

# Beyond Timmons' theorem

- Brandt et al. proved that $\text{mad}(G) < 2.5$ suffices.
- 2.5 is sharp as infinitely many examples with average degree 2.5 have no $I, F$-partition
- The optimal value of $\text{mad}(G)$ implying star 4-colorability is not known.
- It was proved that $s(G) \leq 8$ when $\text{mad}(G) < 3$ and that $s(G) \leq 6$ when $\Delta(G) = 3$ (the latter is sharp).
- No bound on $s(G)$ can be implied by $\text{mad}(G) < 4$. There are examples for this which have average degree tending to 4
- What happens for bounds between 3 and 4 remains open.

# Acyclic 6-choosability

## Lemma 3.8

If $G$ is a planar graph with girth at least 5 and $\delta(G) \geq 2$, then $G$ has a 2-vertex with a 5-neighbor or a 5-face whose incident vertices are four $3^-$ vertices and a 5-vertex.

# 2-distance coloring

## Lemma 4.1

Even cycles are 2-choosable.

## Definition

Given a graph $G$, let $G^2$ be the graph obtained from $G$ by adding edges to join vertices that are distance 2 apart in $G$.

## Lemma 4.2

Fix $k \geq 4$. Among graphs $G$ with $\Delta(G) \leq k$, the following configurations are reducible for the property $\chi_\ell(G^2) \leq k + 1$:

- a $1^-$-vertex
- a 2-thread joining a $(k-1)^-$-vertex and a $(k-2)^-$-vertex
- a cycle of length divisible by 4 composed of 3-threads whose endpoints have degree $k$.

# 2-distance coloring cd.

**Theorem 4.3**

If $\Delta(G) \leq 6$ and $mad(G) < \frac{5}{2}$, then $\chi_\ell(G^2) \leq 7$.

**Theorem**

If $\Delta(G) \geq 6$ and $mad(G) < 2 + \frac{4\Delta(G)-8}{5\Delta(G)+2}$, then $\chi_\ell(G^2) = \Delta(G^2) + 1$.

# Injective coloring

### Definition

A coloring where vertices at distance 2 have distinct colors but adjacent vertices need not is an **injective coloring**.

### Definition

The **injective chromatic number**, written $\chi^i(G)$, is the minimum number of colors needed, and the **injective choice number**, $\chi^i_\ell(G)$, is the least $k$ such that $G$ has an injective $L$-coloring when $L$ is any $k$-uniform list assignment.

### Theorem 4.5

If $\Delta(G) \leq 3$ and $mad(G) < \frac{36}{13}$, then $\chi^i_\ell(G) \leq 5$

# List coloring on planar graphs

## Lemma 4.8

Every normal plane map $G$ has a 3-vertex with a $10^-$-neighbor, or a 4-vertex with a $7^-$-neighbor, or a 5-vertex with two $6^-$-neighbors.

## Theorem 4.9

if $G$ is a planar graph, then $\chi_\ell(G^2) = \begin{cases} \Delta(G^2)+1 & \textit{when } \Delta(G) \leq 5 \\ 7\Delta(G)-7 & \textit{when } \Delta(G) \geq 6 \end{cases}$.

- Index the vertices from $v_n$ to $v_1$ as follows. Having chosen $v_n, \ldots, v_{i+1}\}$, let $G_i = G \setminus \{v_n, \ldots, v_{i+1}\}$. If $\delta(G_i) \leq 3$, then let $v_i$ be a vertex of minimum degree; otherwise let $v$ be a vertex as guaranteed by Lemma 4.8.
- Let $S_i = \{v_1, \ldots, v_i\}$. We choose colors for vertices in the order $v_1, \ldots, v_n$ so that the coloring of $S_i$ satisfies all the constraints in the full graph $G^2$ from pairs of vertices in $S_i$.

# Edge-coloring

**Vizing's Theorem**

$\chi'(G) \leq \Delta(G) + 1$ when $G$ is a graph.

**Definition**

$G$ is **Class 1** if $\chi'(G) = \Delta(G)$, **Class 2** otherwise.

**Definition**

An **edge-critical graph** $G$ is a Class 2 graph such that $\chi'(G \setminus e) = \Delta(G), \forall e \in E(G)$.

**Vizing's Adjacency Lemma**

If $x$ and $y$ are adjacent in an edge-critical graph $G$, then at least $max\{1 + \Delta(G) - d(y), 2\}$ neighbors of $x$ have degree $\Delta(G)$.

# Edge-coloring cd.

**Theorem 5.3**

If $G$ is a graph with $mad(G) < 6$ and $\Delta(G) \geq 8$, then $\chi'(G) = \Delta(G)$.

# List edge-coloring definitions

**Definition**

An **edge-list assignment** $L$ assigns lists of available colors to the edges of agraph $G$.

**Definition**

Given an edge-list assignment $L$, an $L$-**edge-coloring** of $G$ is a proper edge-coloring $\phi$ such that $\phi(e) \in L(e), \forall e \in E(G)$.

**Definition**

A graph $G$ is $k$-**edge-choosable** if $G$ is $L$-edge-colorable whenever each list has size at least $k$.

**Definition**

The **list edge-chromatic number** of $G$, written $\chi'_\ell(G)$, is the least $k$ such that $G$ is $k$-edge-choosable.

# List edge-coloring

## Conjecture 5.5

$\chi'_\ell(G) = \chi'(G)$ for every graph $G$.

## Definition

A $t$-alternating cycle alternates between $t$-vertices and vertices of higher degree.

## Theorem 5.6

If $G$ is a planar graph and $\Delta(G) \geq 9$, then $\chi'_\ell(G) \leq \Delta(G) + 1$.

## Lemma 5.7

If $G$ is a simple plane graph with $(G) \geq 2$, then $G$ contains:

- (C1) an edge $uv$ with $d(u) + d(v) \leq 15$, or
- (C2) a 2-alternating cycle $C$.

# List edge-coloring cd.

**Theorem 5.8**

If $G$ is a plane graph with $\Delta(G) \geq 14$, then $\chi'_\ell(G) = \Delta(G)$.

# Iterated discharging

## Theorem 5.9

If lists on the edges of a bipartite multigraph $G$ satisfy $|L(uv)| \geq max\{d_G(u), d_G(v)\}$ for $uv \in E(G)$, then $G$ has an $L$-edge-coloring.

## Definition

In a multigraph $G$, an $i$-alternating subgraph is a bipartite submultigraph $F$ with parts $U$ and $W$ such that $d_F(u) = d_G(u) \leq i$ when $u \in U$ and $d_G(w) - d_F(w) \leq \Delta(G) - i$ when $w$. Note that cycles in $F$ alternate between $W$ and $i^-$-vertices in $U$.

## Lemma 5.11

$i$-alternating subgraphs are reducible for the property that edge-choosability equals maximum degree.

# Iterated discharging cd.

**Theorem 5.12**

If $mad(G) \leq \sqrt{2\Delta(G)} - 1$, then $\chi'_\ell(G) = \Delta(G)$.