Woodall's conjecture

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- E_G: edges of graph G
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- Sink-set: transposition of source-set

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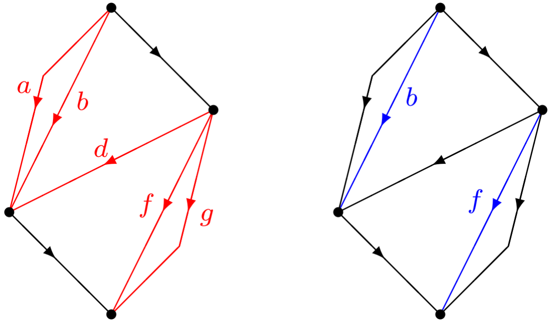

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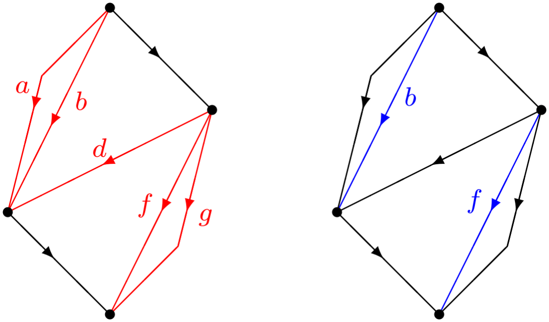

A cut C is **peripheral** if $C = \Delta^+(X)$ and X is minimal source-set or $C = \Delta^-(Y)$ and Y is minimal sink-set.

Notation

- Strengthening set: set B of edges with following property: $\forall u, v \in V_G$ there exist a path from u to v in $G' = (G, (E - B) \cup B^{-1})$

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A collection D of joins is **disjoint** if its elements are pairwise disjoint. Also, for any graph G, we define $\nu(G)$ and $\tau(G)$:

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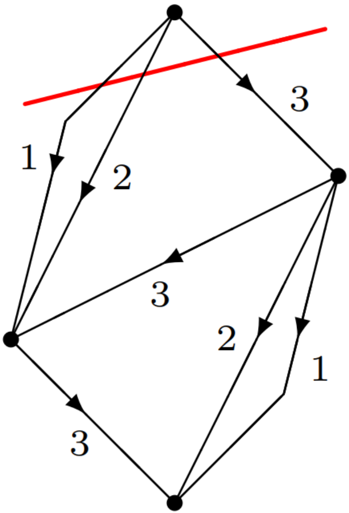
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We have the following relation:

Lemma

For any graph G, we have $\nu(G) \leqslant \tau(G)$

Woodall's conjecture

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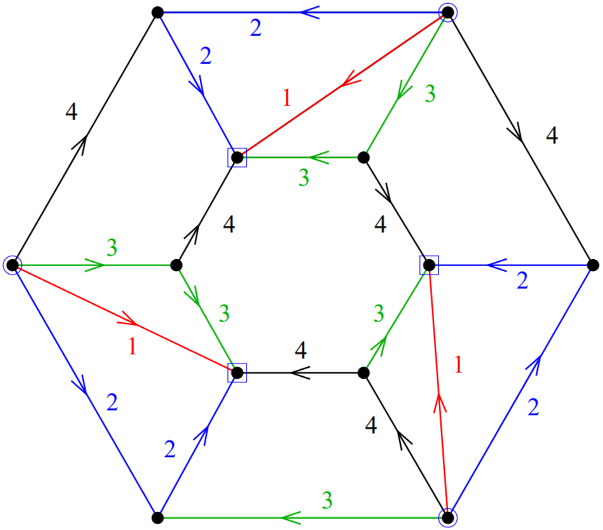
Woodall's conjecture

Let $G = (V, E)$ and $k \geqslant 2$. Then E can be partitioned into k strengthening sets if and only if each directed cut has size at least k.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Lucchesi-Younger theorem

Any maximum disjoint collection of cuts has the same cardinality as a minimum join.

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Capacitated generalization of Woodall's conjecture

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Capacitated Woodall

For every graph G and any $c \in \mathbb{N}^E$, we have $\nu(G, c) = \tau(G, c)$

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Capacities versus active edges

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If $c(e) \geqslant 1$, we can simulate that by $c(e)$ copies of e in paralell.

But if $c(e) = 0$, the deletion of an edge e may create new cuts, that are not of the form $C - \{e\}$ where C is one of the original cuts.

So we will replace the deletion of an edge by the **deactivation** of the edge.

A **graaph** is a pair (G, A) where G is a graph and A is a part of E_G. The edges in A are **active** and those in $E_G - A$ are inactive.

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Active edges Woodall's conjecture

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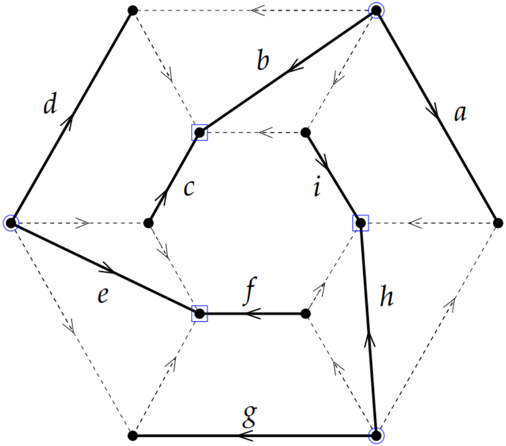
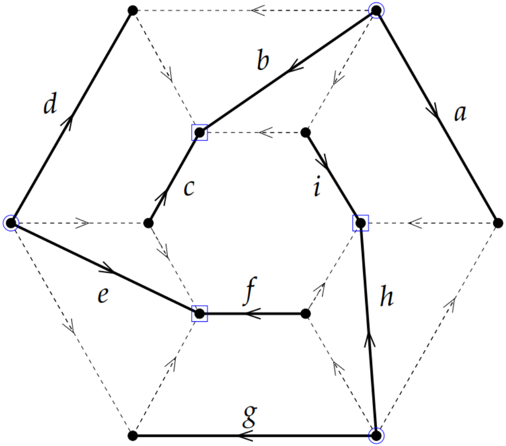
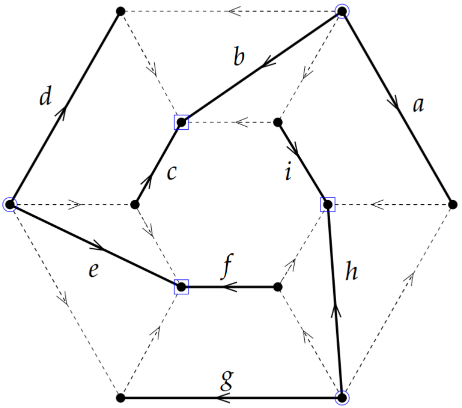
Schrijver's counterexample





$\nu(G, A) = 1$ and $\tau(G, A) = 2$



a	b	c	d	e	f	g	h	i
J	$\tilde{J}$	J	J	$\tilde{J}$	J	J	$\tilde{J}$	J
J	$\tilde{J}$	J	J	$\tilde{J}$	J	$\tilde{J}$	J	$\tilde{J}$
J	$\tilde{J}$	J	$\tilde{J}$	J	$\tilde{J}$	J	$\tilde{J}$	J
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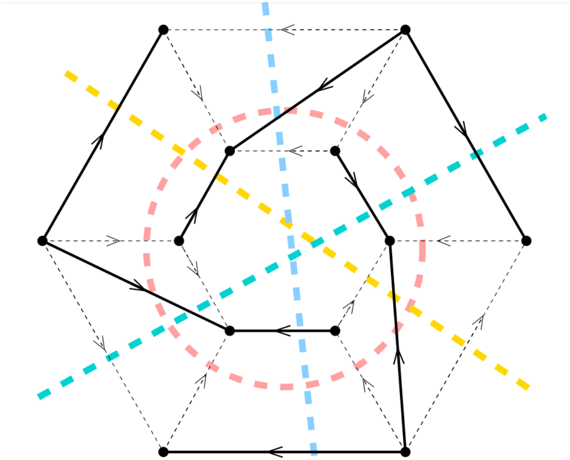
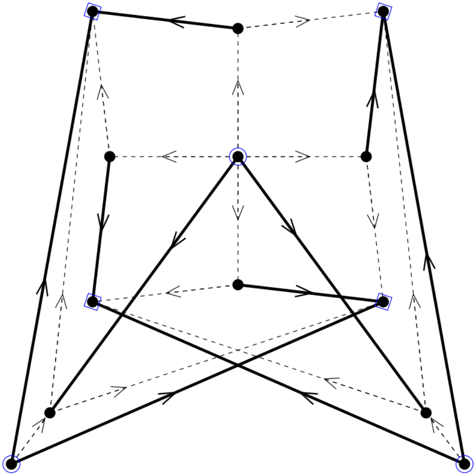
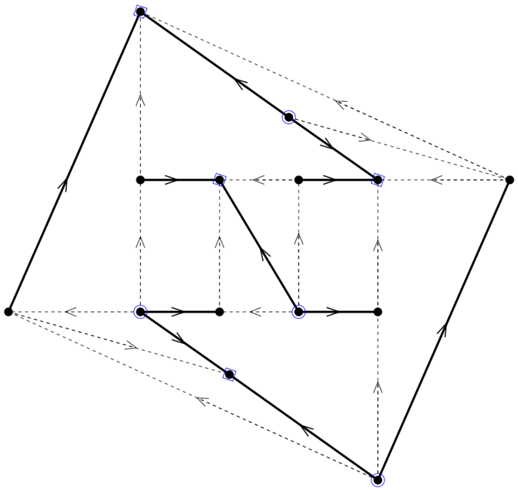


Cornuéjols–Guenin counterexample 1



Cornu´ejols–Guenin counterexample 2



Partial results

Shrijver, Feofiloff

When the graph is acyclic and every source is joined to every sink by a directed path, generalized conjecture (with active edges and capacities) is **true**.

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$\tau = 2$

Proven to be true if $\tau = 2$ by Seymour and M.DeVos.

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Conjecture $\tau = 3$

There exists a fixed integer $k \geqslant 3$ so that every digraph with all directed cuts of size $\geqslant k$ contains three pairwise disjoint dijoins.

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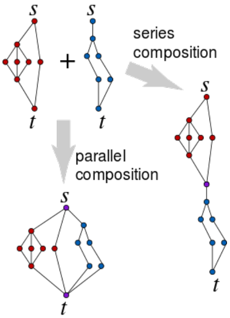
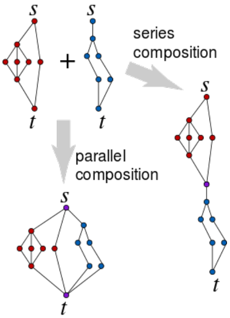
A two-terminal series–parallel graph (TTSPG) is a graph that may be constructed by a sequence of series and parallel compositions starting from a set of copies of a single-edge graph K_2 with assigned terminals.

Series parallel graph





Lee and Wakabayashi

Woodall's conjecture is true for series-parallel digraphs.

Thanks!

Image sources:
https://www.ime.usp.br/~pf/dijoins/woodall/survey1-en.pdf
https://en.wikipedia.org/wiki/Series%E2%80%93parallel_graph