The Fixing Block Method in Combinatorics on Words

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Problem 1.2.

Is there a sequence over a 6 letter alphabet which is non-repetitive up to mod 4?

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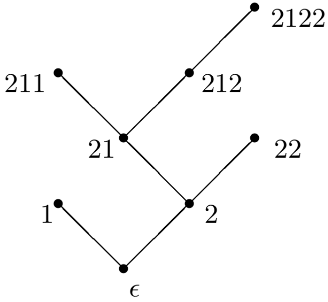

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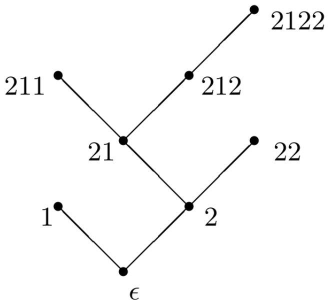

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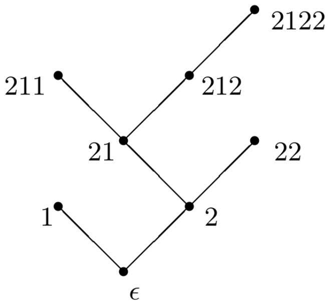

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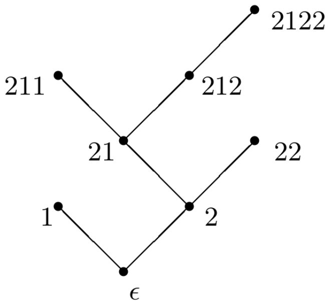

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- $\hat{v}$ is an upper cover of v if $v < \hat{v}$, but there is no $z \in L$ such that $v < z < \hat{v}$.
- Given words $u \leqslant v$ in L, the closed interval $[u, v]$ is the set $\{w \in L : u \leqslant w \leqslant v\}$.
  $[u, \infty] = \{w \in L : u \leqslant w\}$

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- $[u, \infty] \setminus [v, \infty]$ is finite.

In this case any path in L from u to ∞ must traverse the vertices of $[u, \hat{v}]$.

Bottlenecks

We refer to the set $B(u, v) = [u, \infty] \backslash [\hat{v}, \infty]$ as a bottleneck with **core** $[u, v]$. If $[v, \infty]$ is finite, let $B(u, v) = [u, \infty]$.

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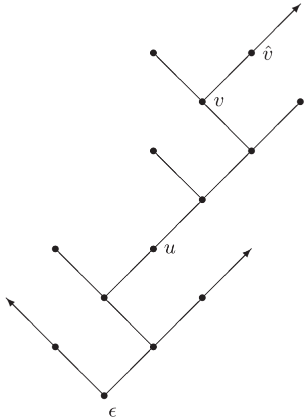

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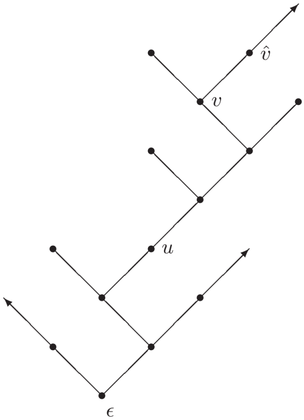

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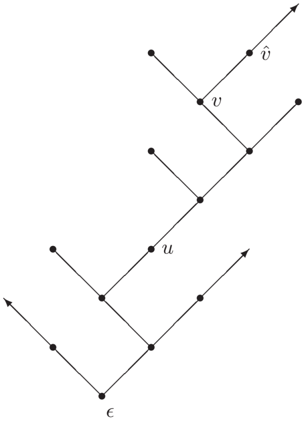

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Suppose that $w \leqslant y$ are elements of $[u, v]$. It follows that at most one cover of y has an infinite extension and we can form a bottleneck $B(w, y)$.

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Inequality 1.

Our approach is to show that long bottlenecks in L must occur far out.

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Here N_0 is some constant, while f is eventually increasing and unbounded. Suppose now that Inequality 1 holds. The existence of such an inequality gives information about the structure of L.

Theorems involving Inequality 1.

Let $g > N_0$ be a function such that for any non-negative integer M we have $f(x) > M$ whenever $x > g(M)$.

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Word $u \in L$ is a prefix of infinitely many words in L if and only if L contains a word $v > u$, with $|v| - |u| = g(|u|)$.

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Proof.

It suffices to prove the if direction. Suppose that L contains a word $v > u$ with $|v| - |u| = g(|u|)$, but that $[u, \infty]$ is finite. Since $[u, \infty]$ is finite, $B = B(u, v) = [u, \infty]$ is a bottleneck. We have $\iota(B) = |u|$, and $|B| = |v| - |u| + 1 > g(|u|)$. By Inequality 1:

$$|u| = \iota(B) \geqslant f(|B|) = f(|v| - |u| + 1) > |u|$$

This is a contradiction. □

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Suppose that L is infinite but not perfect. There is some word $u \in L$ with exactly one infinite extension. Let $v > u$ be any finite prefix of the unique infinite extension of u. There will correspond a bottleneck $B(u, v)$ of index $|u|$ and length $|v| - |u| + 1$. Because $|u|$ is fixed, but $|v|$ can be made arbitrarily large, this violates Inequality 1. □

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Theorem 3.4.

The set of non-repetitive words over $\{1, 2, 3\}$ of length n grows exponentially.

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- $w3 = 231213231213 \notin L$, suffix $w_3 = w$ has period 6.

For each letter $s \in \Sigma$ such that $ws \notin L$, w must have a periodic suffix which is 'near square'. A maximal non-repetitive word has several such suffixes. We call them **fixing blocks**.

Fixing blocks are suffixes of non-repetitive words of the form ycy where y is a word of L, c a letter of Σ. The **period** of such a fixing block is $l = |yc|$.

Fixing Blocks Inequalities

In a word with several periods, interference patterns can arise. For example, if an infinite periodic word has periods α and β, it also has period $\gamma = gcd(\alpha, \beta)$. A more complicated interaction happens when two distinct non-repetitive words v_1 and v_2 end in fixing blocks, and we consider the possible interference of these blocks in the common prefix $v_1 \wedge v_2$.

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Lemma 4.1.

For $i = 1, 2$, let v_i be a word of L with a fixing block suffix of period l_i. Suppose that:

1. $l_2 \geqslant l_1$
2. Not both $l_1 = l_2$ and $|v_1| = |v_2|$
3. $l_2 > |v_2| - |v_1 \wedge v_2|$

Then:

$$l_2 \geqslant 2l_1 - (|v_1| - |v_1 \wedge v_2|)$$

Fixing Blocks Inequalities

Using induction we can already show that reasonably long fixing blocks occur:

Corollary 4.2.

For $i = 1, \ldots, r$, let v_i be a word in L with a fixing block of period l_i. Suppose that for $i = 1, \ldots, r-1$ we have:

1. $l_{i+1} \geqslant l_i$
2. Not both $l_i = l_{i+1}$ and $|v_i| = |v_{i+1}|$
3. $l_{i+1} > |v_{i+1}| - |v_i \wedge v_{i+1}|$

Then:

$$l_r \geqslant 2^{r-1} l_1 - \sum_{j=1}^{r-1} 2^{r-1-j}(|v_j| - |v_j \wedge v_{j+1}|)$$

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If v is a word of L with exactly d upper covers, $d < |\Sigma|$, then v ends in a fixing block of period $\geqslant 2^{|\Sigma|-d-1}$.

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Similar lemma can be shown in a language L of words non-repetitive up to mod r.

Lemma 4.4.

If v is a word of L with exactly d upper covers, $d < |\Sigma|$, then v ends in a fixing block of period $\geqslant 2^{(|\Sigma|-d-1)/r}$.

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Regular bottlenecks

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Any bottleneck B contains a regular bottleneck of length $|B|$. (Just take the length B suffix of the longest maximal word in B.)

Inductive Lemmas

Lemma 5.1.

Suppose there exist numbers m and $\alpha, \alpha > 5$, such that every regular bottleneck of length at least m contains a word with fixing block period at least αm. Then each bottleneck of length at least $4m$ contains a word with a fixing block period of at least $4m(2\alpha - 5)$.

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Induction gives the following:

Lemma 5.2.

If every regular bottleneck of length at least m contains a word with a fixing block period at least αm, for some $\alpha > 5$, then each bottleneck of length $\geqslant 4^n m$ must contain a word with a fixing block period of at least $4^n m(2^n(\alpha - 5))$.

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Suppose we can find such m and α. Then Lemma 5.2. gives an inequality of the form of Inequality 1.

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As noted in Section 3, this gives the interesting result that for such Σ, the set of infinite non-repetitive words over Σ is perfect.

Inductive Lemmas

For the rest of this section, let L be the language of words over an alphabet Σ which are non-repetitive up to mod r. Let s be chosen so that

$$(2^{s-1} - (r(s-1)+1)) > 0$$

Let α be chosen so that

$$\alpha > 2(r(s-1)+1)$$

and

$$\alpha > \frac{2(r(s-1)+1)(2^{s-2}-1)}{2^{s-1}-(r(s-1)+1)}$$

Inductive Lemmas

Lemma 5.3.

Suppose that every regular bottleneck of length m has a vertex with a fixing block of period at least αm. Then every regular bottleneck of length at least $(r(s-1)+1)m$ has a vertex with a fixing block of period at least $\alpha(r(s-1)+1)m$.

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Proof sketch.

Let B be a regular bottleneck of length $(r(s-1)+1)m$. Divide the core of B into $r(s-1)+1$ disjoint paths, each of length m. Each of these paths is the core of a bottleneck of length m, therefore we can find $r(s-1)+1$ disjoint regular bottlenecks of length m, so $r(s-1)+1$ distinct vertices in B, each having a fixing block of period at least αm. We apply Lemma from previous section and, by the choice of α, we have

$$I_{r(s-1)+1} \geqslant \alpha(r(s-1)+1)m$$

Inductive Lemmas

If we can find α and m as in Lemma 5.3. we can get an inequality of form Inequality 1. This will enable us to show that L is perfect. However, it requires s and α to be rather large. An induction can be started with $r = 4, s = 6, \alpha = 630/13, m = 1$, if $|\Sigma| \geqslant 35$; this follows from Lemma 4.4. We get the following results:

Inductive Lemmas

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Theorem 5.5.

The set of infinite words over an alphabet Σ which are non-repetitive up to mod r is perfect if $|\Sigma|$ is sufficiently large.

'Sufficiently large' can be replaced by a constructive condition based on first choosing s, then α, then Σ.