PROBLEMS AND RESULTS ON 3-CHROMATIC HYPERGRAPHS AND SOME RELATED QUESTIONS

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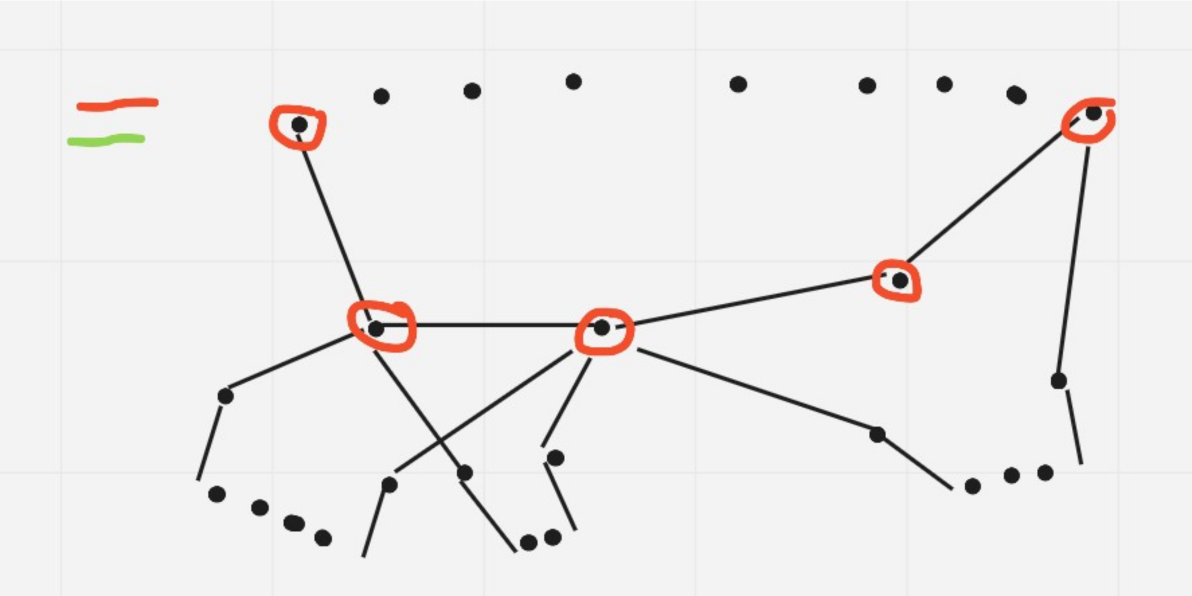
Abstract

- Authors in this work aim to establish various bounds and constraints on hypergraphs which are k-chromatic.
- *Hypergraph* is a graph where an edge don't have to link exactly two vertices. Hypergraph is called simple, when none two of his edges has more then one common point, and is called clique when each two of his edges has at least one common point. Hyper graph is *r-uniform* when each of its edges contains exactly *r* points.
- *Chromatic number* is a smallest number k such that you can color points of the graph using k colors in the way that no edge is monochromatic.
- Main part of the work involves around the impact that being clique or simple has on 3-chromatic hypergraph structure. The main reason why those two things are connected is following trivial observation: *If a hypergraph has chromatic number > 3 then it has two edges with exactly one common point.*

Trivial observation

If a hypergraph has chromatic number $\geqslant 3$ *then it has two edges with exactly one common point.*



Theorem 1

Let $m_k(r)$ be the minimum number of edges of a $(k+1)$-chromatic r-uniform hypergraph. It is known [5], [9]

$$\frac{r}{r+2}2^{r-1} \leqslant m_2(r) \leqslant r^2 2^r .$$

Let $n_k^*(r)$, $m_k^*(r)$ denote the minimum number of points and edges in a $(k+1)$-chromatic r-uniform *simple* hypergraph. We shall prove

Theorem 1.

$$\lim_{r\to\infty} \sqrt[r]{n_k^*(r)} = k ,$$

$$\lim_{r\to\infty} \sqrt[r]{m_k^*(r)} = k^2 .$$

Theorm 1'

Theorem 1'. *Let* $s \geqslant 2,\ r \geqslant 2,\ k \geqslant 2;\ n = 4 \cdot 20^{s-1} r^{3s-2} \cdot$ $\cdot\, k^{(s-1)(r+1)},\ m = 4 \cdot 20^{s} \cdot r^{3s-2} \cdot k^{s(r+1)},\ d = 20r^{2} \cdot k^{r-1}.$

Then there exists an r-uniform hypergraph H on $k \cdot n$ points with at most m edges and with degrees $\leqslant d$ which does not contain any circuits of length $\leqslant s$ and in which each set of n points contains an edge.

This hypergraph is, obviously, at least $(k+1)$-chromatic.

Construction

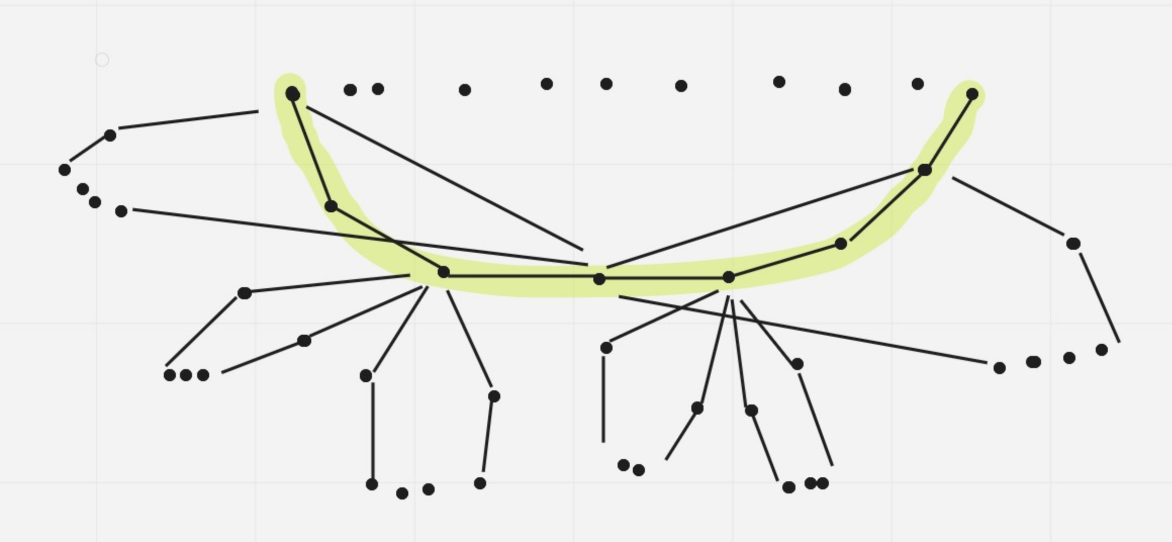
$$x_{p+1} \leqslant x_p \left(1 - \frac{1}{20k^r}\right). \qquad (2)$$

Suppose we know that if $p < m$ then (2) holds. Then

$$x_m \leqslant x_0 \left(1 - \frac{1}{20k^r}\right)^m < 2^{kn} \cdot e^{\frac{-m}{20k^r}} < e^{kn - \frac{m}{20k^r}} = 1$$

Theorem 2

Theorem 2. *A* $(k+1)$*-chromatic* r*-uniform hypergraph contains an edge which is intersected by at least* $k^{r-1}/4$ *other edges. Thus, the valency of at least one vertex is* $> k^{r-1}/4r$.



Lemma to Theorem 2

Lemma. *Let* G *be a (finite) graph with maximum degree* d *and vertices* $v_1, \ldots, v_n$. *Let us associate an event* A_i *with* v_i $(i = 1, \ldots, n)$ *and suppose that* A_i *is independent of the set*

$$\{A_j : (v_i, v_j) \in E(G)\} .$$

Also suppose

$$P(A_i) \leq \frac{1}{4d} . \tag{3}$$

Then

$$P(\bar{A}_1 \ldots \bar{A}_n) > 0 . \tag{4}$$

Proof. We prove more, namely that

$$(5) \qquad P(A_1 \mid \bar{A}_2 \dots \bar{A}_n) \leqslant \frac{1}{2d}.$$

$$P(A_1 \mid \bar{A}_2 \dots \bar{A}_n) = \frac{P(A_1 \bar{A}_2 \dots \bar{A}_q \mid \bar{A}_{q+1} \dots \bar{A}_n)}{P(\bar{A}_2 \dots \bar{A}_q \mid \bar{A}_{q+1} \dots \bar{A}_n)}.$$

$$P(A_1 \bar{A}_2 \dots \bar{A}_q \mid \bar{A}_{q+1} \dots \bar{A}_n) \leqslant$$
$$\leqslant P(A_1 \mid \bar{A}_{q+1} \dots \bar{A}_n) = P(A_1) \leqslant \frac{1}{4d},$$

$$P(\bar{A}_2 \dots \bar{A}_q \mid \bar{A}_{q+1} \dots \bar{A}_n) =$$
$$= 1 - P(A_2 + \dots + A_q \mid \bar{A}_{q+1} \dots \bar{A}_n) \geqslant$$
$$\geqslant 1 - \sum_{i=2}^{q} P(A_i \mid \bar{A}_{q+1} \dots \bar{A}_n) \geqslant 1 - (q-1)\frac{1}{2d} \geqslant \frac{1}{2}$$

Corollary 1

Corollary 1. *If each point of an r-uniform hypergraph H has degree $\leqslant k^{r-1}/4r$ then the chromatic number of H is $\leqslant k$.*

Corollary 2

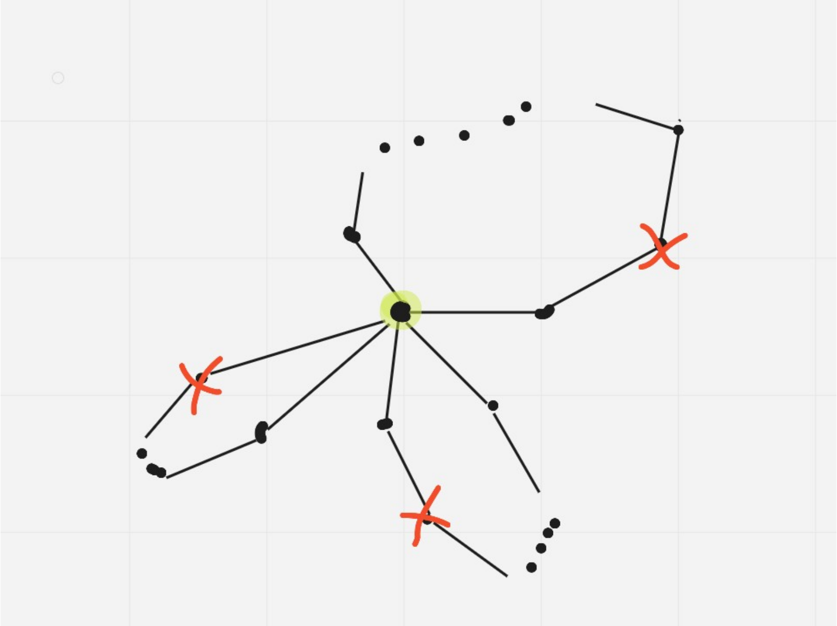
Corollary 2. *If* H *is a simple* $(k+1)$*-chromatic* r*-uniform hypergraph then* $|V(H)| > c \cdot k^{r-1}$.

Theorem 3

Theorem 3. *If each edge of an r-uniform hypergraph H meets at most $k^{r-1}/4(k-1)^r$ other edges then the vertices of H can be k-colored in such a way that each color meets each edge. We also prove the stronger version of* Strauss' *conjecture* (Theorem 4.)

Theorem 5

Theorem 5. *If H is a simple $(k+1)$-chromatic r-uniform hypergraph then it contains at least $k^{r-2}/4(r-1)$ points with degree $\geq k^{r-2}/4(r-1)$.*



Corollary 1

Corollary 1. *A* $(k+1)$*-chromatic* r*-uniform simple hypergraph cannot be covered by less than* $k^{r-2}/4(r-1)$ *points.*

Corollary 2

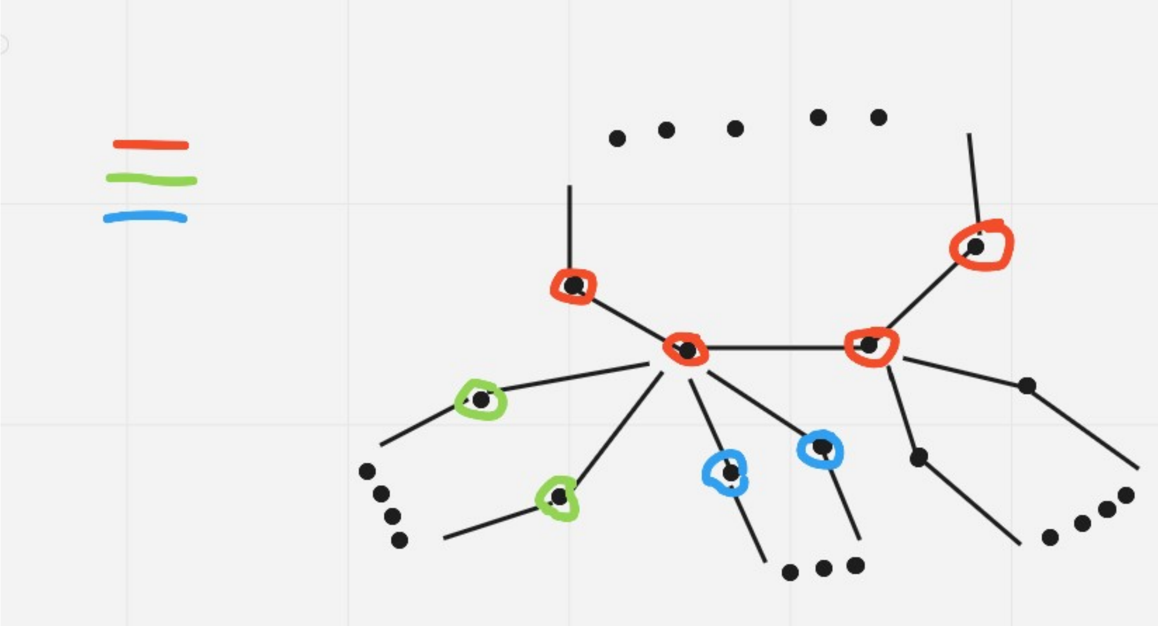
Corollary 2. *A* $(k+1)$*-chromatic* *r-uniform simple hypergraph contains at least* $k^{r-2}/4r(r-1)$ *disjoint edges.*

Corollary 3

Corollary 3. *A* $(k+1)$*-chromatic* r*-uniform simple hypergraphs has at least* $k^{2(r-2)}/16r(r-1)^2$ *edges.*

Fact about cliques

Consider now *r*-uniform cliques. Obviously, a clique can have chromatic number 2 or 3 only; we are interested in those with chromatic number



Theorem 7

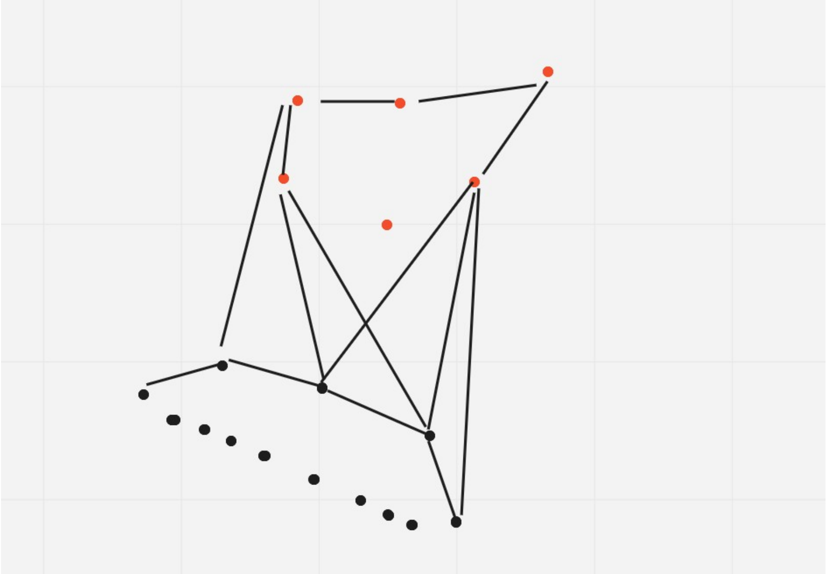
Somewhat surprisingly, there are only finitely many 3-chromatic r-uniform cliques for a given r, so we may ask for the maximum number $M(r)$ of edges in them. We have the inequalities

Theorem 7. $r!(e-1) \leqslant M(r) \leqslant r^r$.

To obtain the upper bound we only use the fact that the edges of a 3-chromatic r-uniform hypergraph cannot be represented by $r-1$ points.

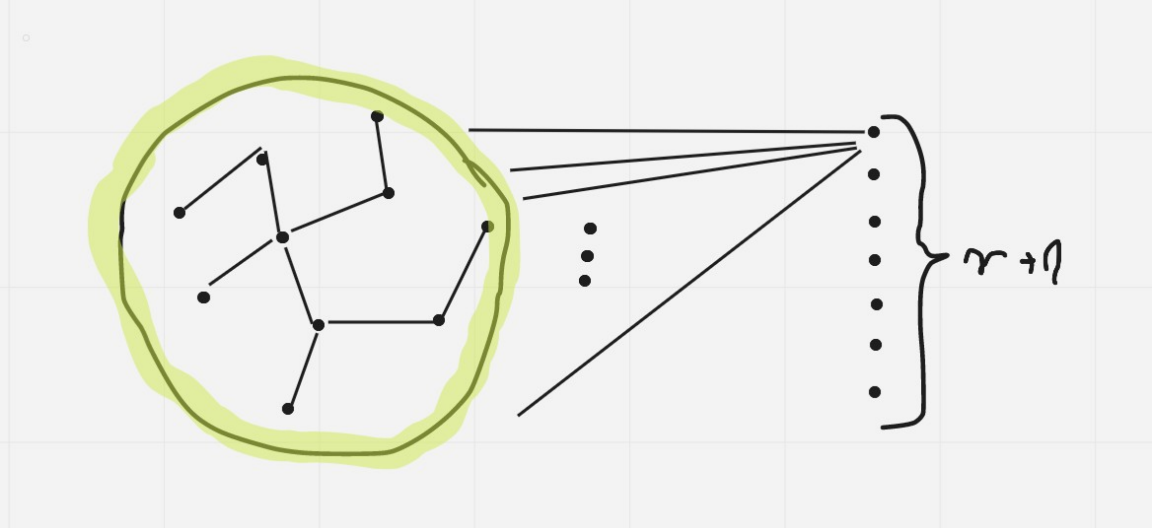
Upper bound

Let us define $x_1, \ldots, x_r$ inductively as follows. Suppose $x_1, \ldots, x_i$ are defined in such a way that more than r^{r-i} edges contain all of them.



Lower bound

(c) Let H be a 3-chromatic r-uniform clique. Let $T \cap V(H) = \phi$, $|T| = r + 1$ and define H' to consist of T and all $(r + 1)$-tuples of the form $E \cup \{t\}$, $E \in H$, $t \in T$. Then H' is an $(r + 1)$-uniform 3-chromatic clique.



Sources:

- PROBLEMS AND RESULTS ON 3-CHROMATIC HYPERGRAPHS AND SOME RELATED QUESTIONS - P. ERDOS - L. LOVÁSZ