Weak degeneracy of graphs

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Theoretical Computer Science

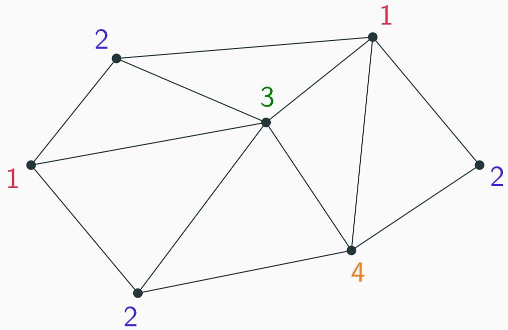
Chromatic number

Coloring

Function $\phi : V(G) \to C$ is a **coloring** of G if:

- $\phi(u) \neq \phi(v)$ for each $uv \in E(G)$

$\chi(G)$ = minimum number of colors $|C|$ required to color vertices of G



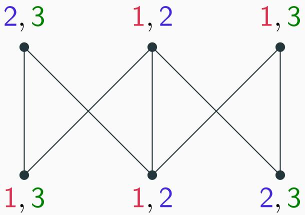
List chromatic number

List coloring

Each vertex $v \in V(G)$ is assigned a list L_v. ϕ is an L-**coloring** of G if:

- $\phi(u) \in L_u$ for each $u \in V(G)$
- $\phi(u) \neq \phi(v)$ for each $uv \in E(G)$

$\chi_L(G)$ = minimum k such that G has an L-coloring whenever each $|L_v| \geq k$



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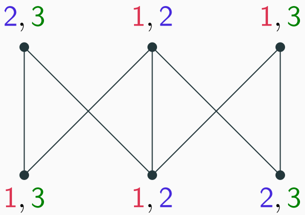


$$\chi(G) \leq \chi_L(G)$$

DP chromatic number

DP coloring

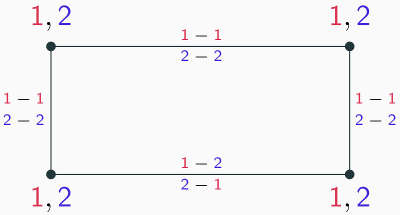
Each vertex $v \in V(G)$ is assigned a list L_v.

Each edge $uv \in E(G)$ is assigned a matching C_{uv} from L_u to L_v.

ϕ is an (L, C)-**coloring** of G if:

- $\phi(u) \in L_u$ for each $u \in V(G)$
- $\phi(u)\phi(v) \notin C_{uv}$ for each $uv \in E(G)$

$\chi_{DP}(G)$ = minimum k such that G has an (L, C)-coloring whenever each $|L_v| \geq k$



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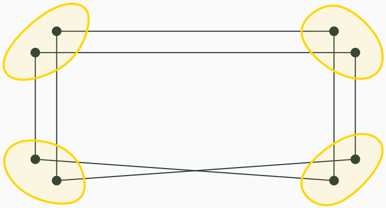

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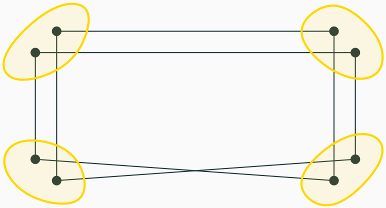
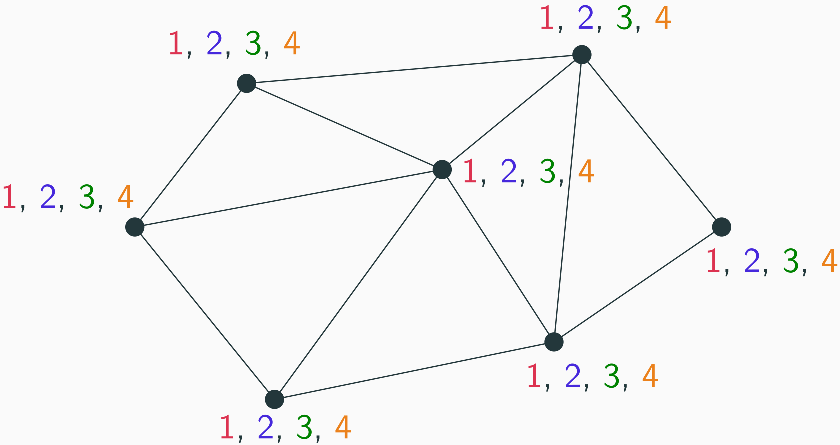


$$\chi(G) \leq \chi_{\mathrm{L}}(G) \leq \chi_{\mathrm{DP}}(G)$$

Greedy (DP)-coloring



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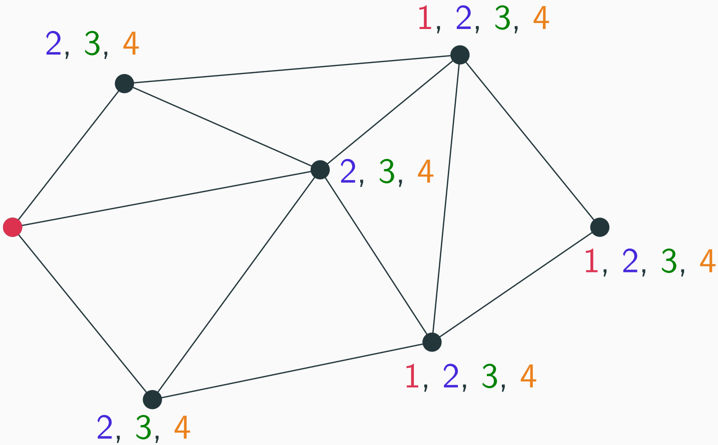

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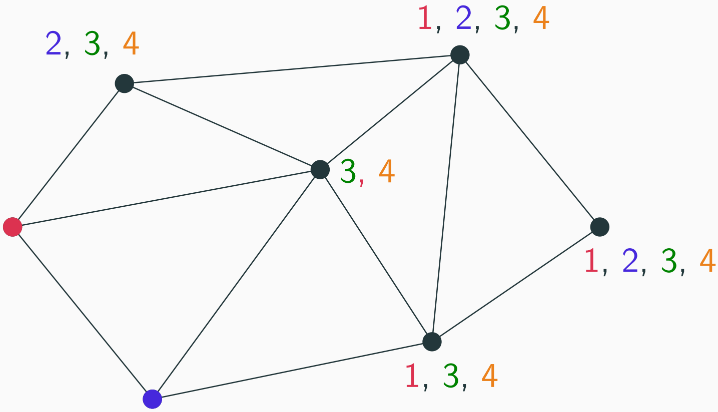

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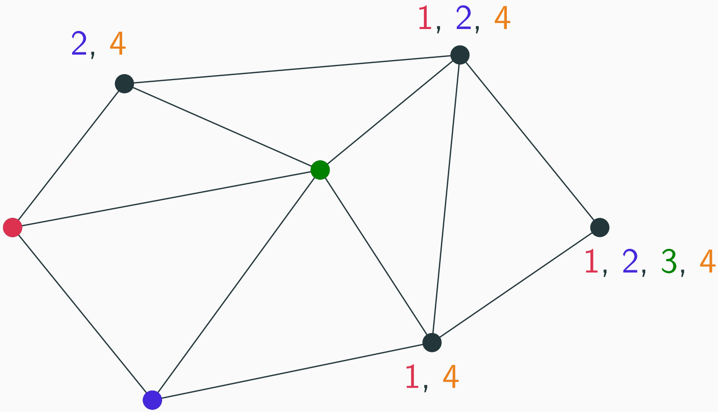

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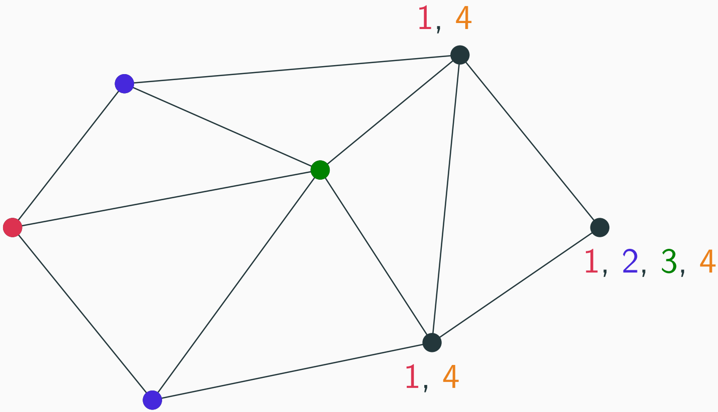

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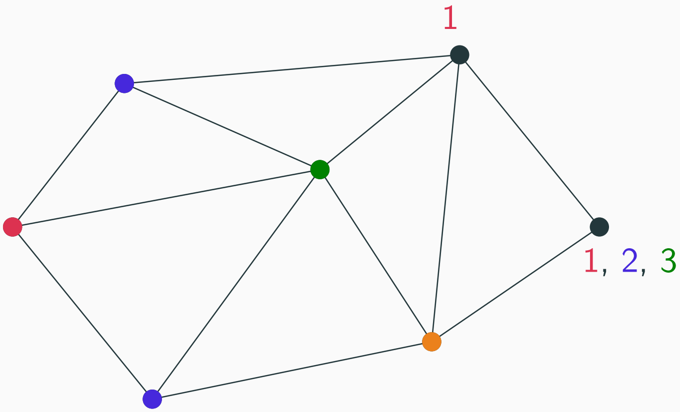

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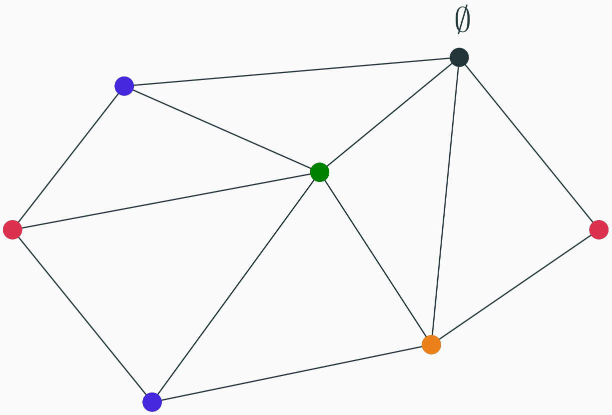

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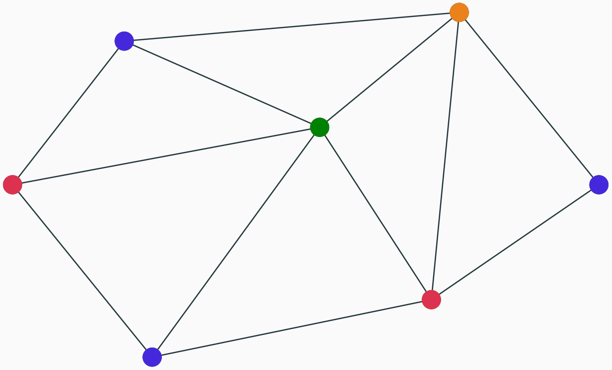


Degeneracy

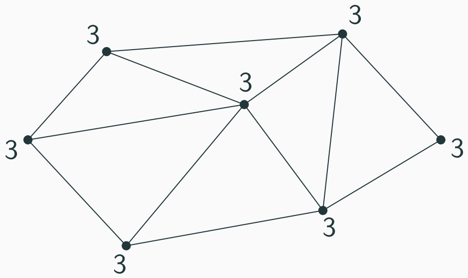
Delete operation

Let $f : V(G) \to \mathbb{N}$ be a function, and let $u \in V(G)$.

Operation **Delete**(G, f, u) outputs graph $G \setminus \{u\}$ and function $f' : G \setminus \{u\} \to \mathbb{N}$:

$$f'(v) = \begin{cases} f(v) - 1 & \text{if } uv \in E(G); \\ f(v) & \text{otherwise.} \end{cases}$$

Operation is **legal** if f' is non-negative.



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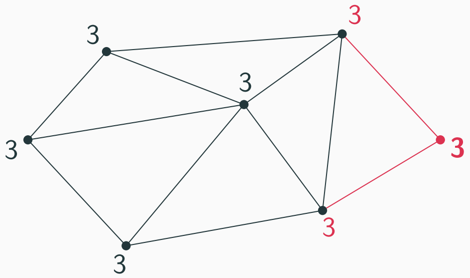

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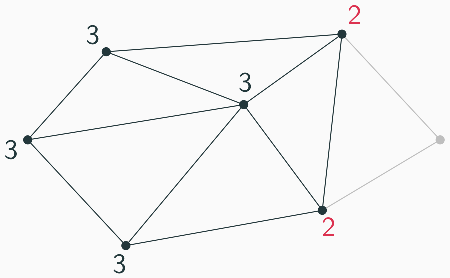

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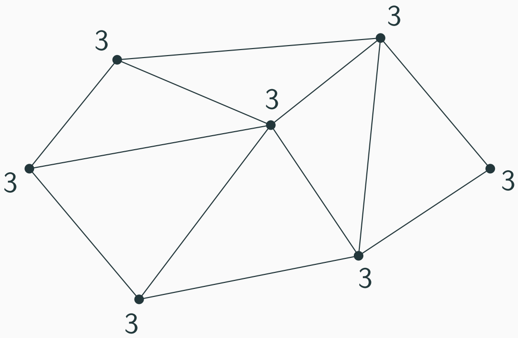
Graph G is f-**degenerate** if it is possible to remove all vertices of G using a sequence of legal Delete operations, starting with function f.

$\mathrm{d}(G)$ = minimum d such that G is d-degenerate (for constant function d)



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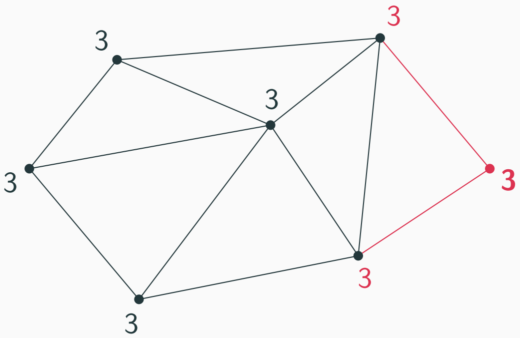

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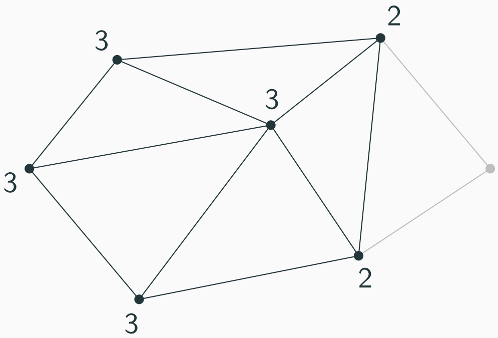

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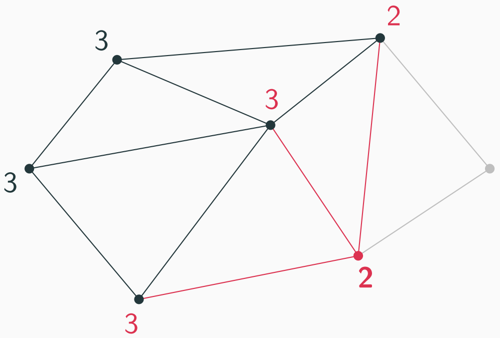

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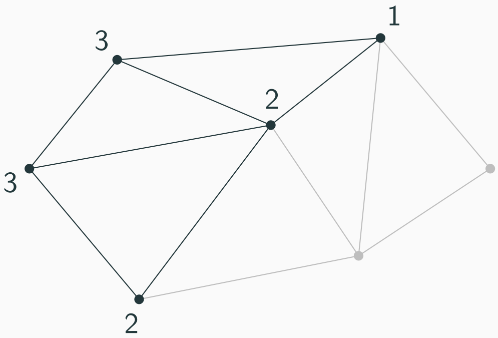

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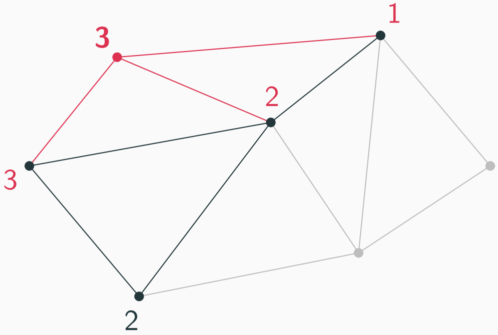

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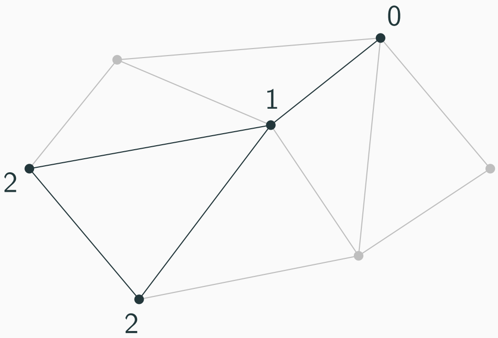

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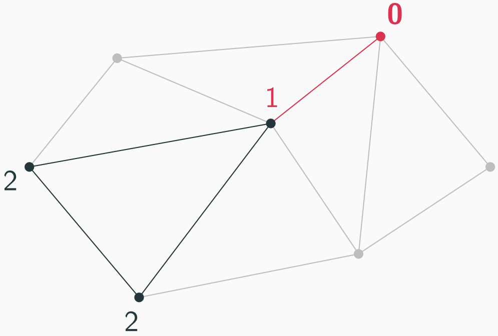

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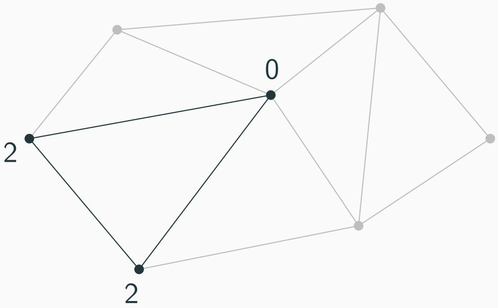

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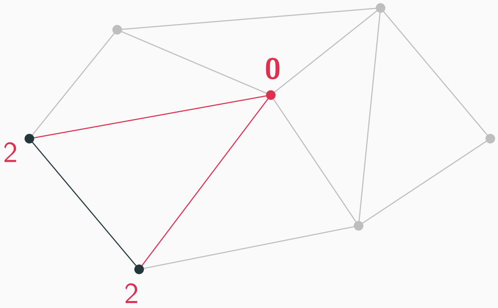

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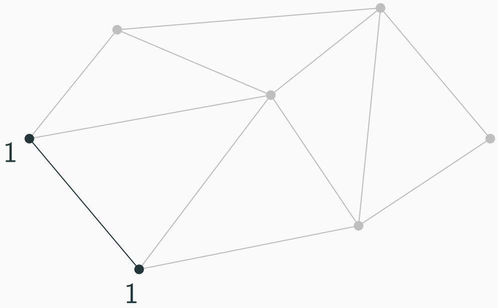

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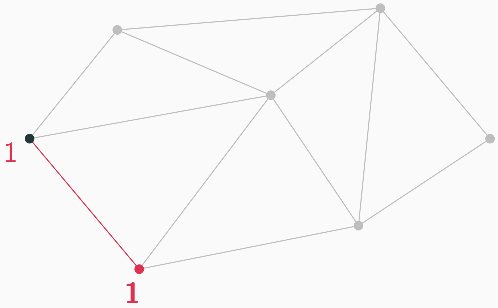

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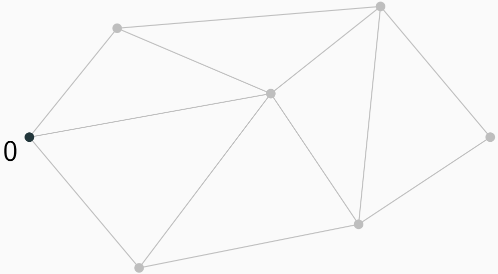

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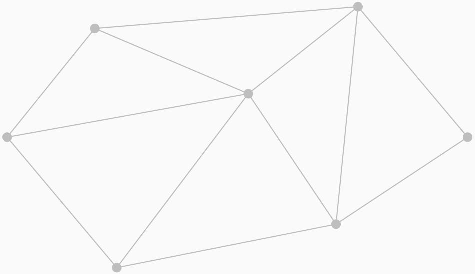

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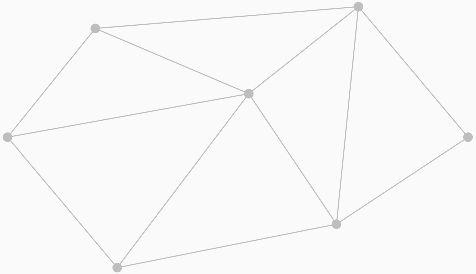
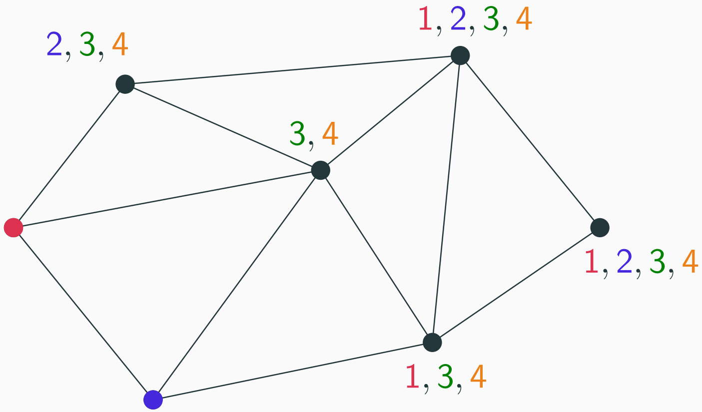


$$\chi(G) \leq \chi_{\mathrm{L}}(G) \leq \chi_{\mathrm{DP}}(G) \leq \mathrm{col}(G) = \mathrm{d}(G) + 1$$

Saving a color



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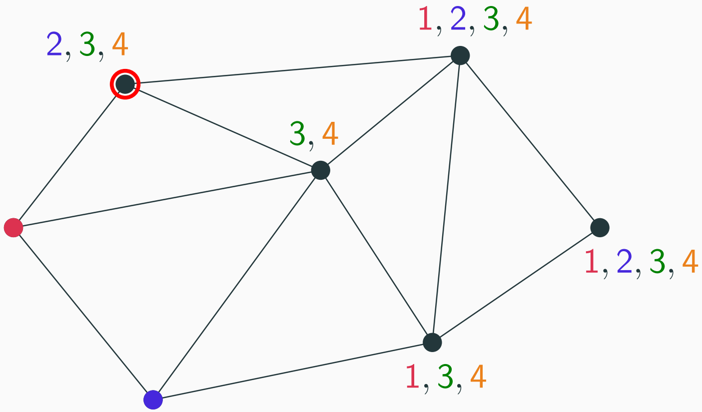

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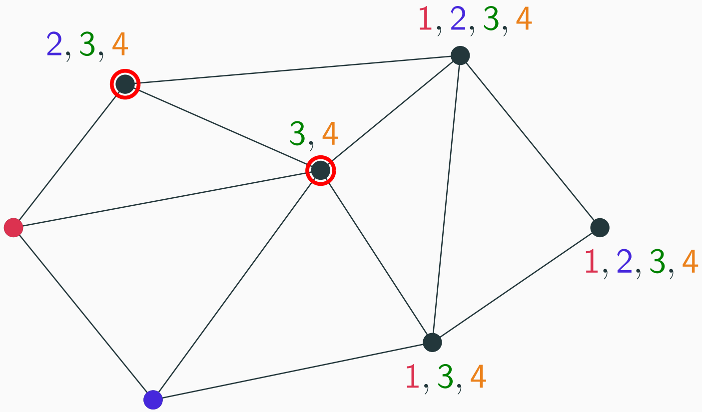

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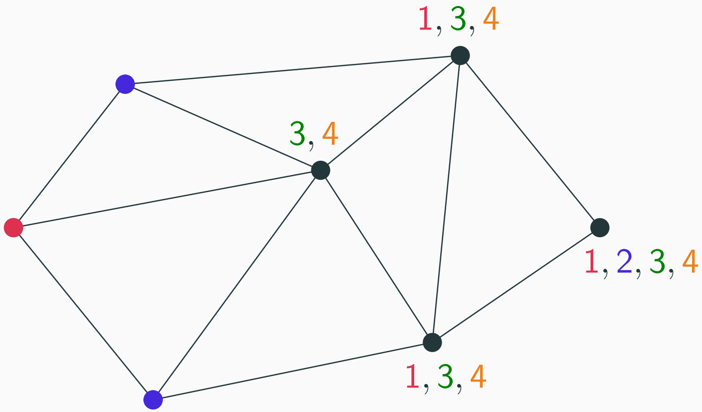

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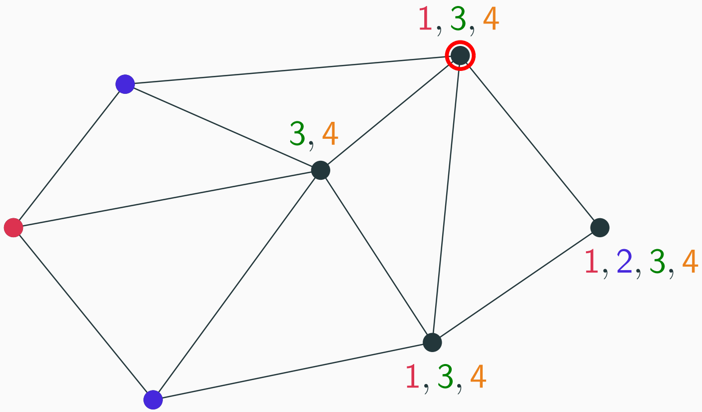

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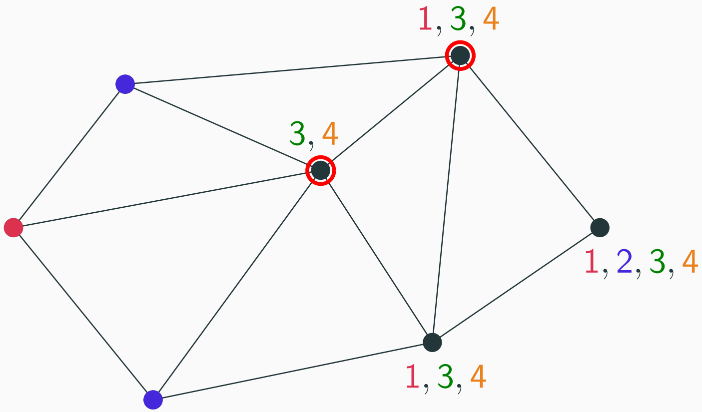

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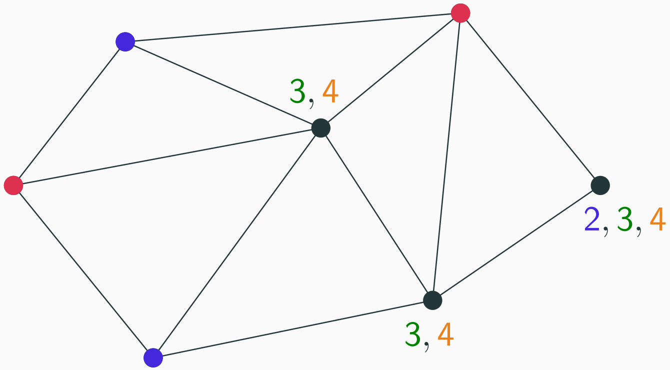


Weak degeneracy

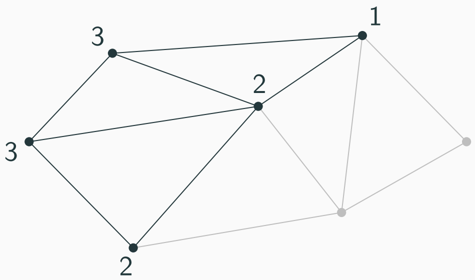
DelSave operation

Let $f : V(G) \to \mathbb{N}$ be a function, and let $u, w \in V(G)$.

Operation **DelSave**(G, f, u, w) outputs graph $G \setminus \{u\}$ and function $f' : G \setminus \{u\} \to \mathbb{N}$:

$$f'(v) = \begin{cases} f(v) - 1 & \text{if } uv \in E(G) \text{ and } v \neq w; \\ f(v) & \text{otherwise.} \end{cases}$$

Operation is **legal** if $f(u) > f(w)$ and f' is non-negative.



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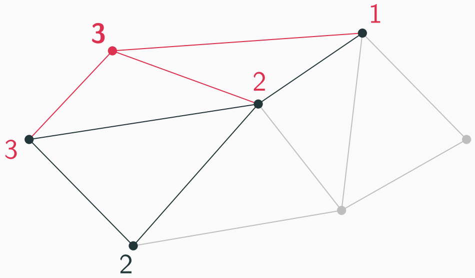

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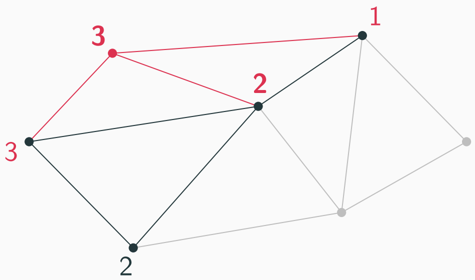

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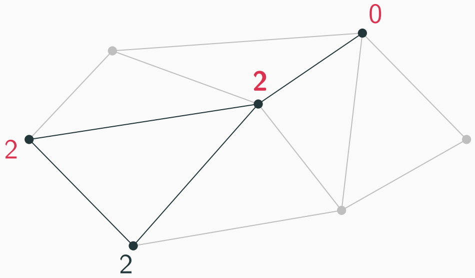

Weak degeneracy

Graph G is **weakly f-degenerate** if it is possible to remove all vertices of G using a sequence of legal **Delete** and **DelSave** operations, starting with function f.

$\mathrm{wd}(G)$ = minimum d such that G is weakly d-degenerate

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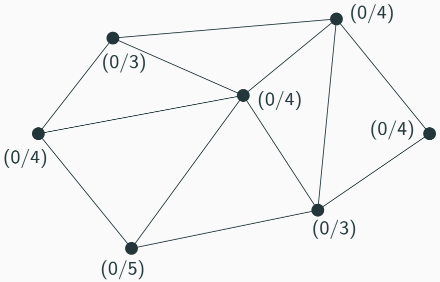
$$\chi(G) \leq \chi_{\mathrm{L}}(G) \leq \chi_{\mathrm{DP}}(G) \leq \mathrm{wd}(G) + 1 \leq \mathrm{d}(G) + 1$$

DP-painting game

Let $f : V(G) \to \mathbb{N}$ be a function. The **DP-painting** game on (G_0, f) is played by **Lister** and **Painter**. The i-th round proceeds as follows:



Online DP-coloring

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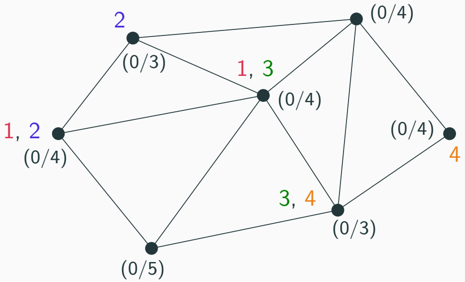
- Lister picks $L_i(u)$ for each $u \in V(G_i)$ and $C_{i,uv}$ for each $uv \in E(G_i)$.



DP-painting game

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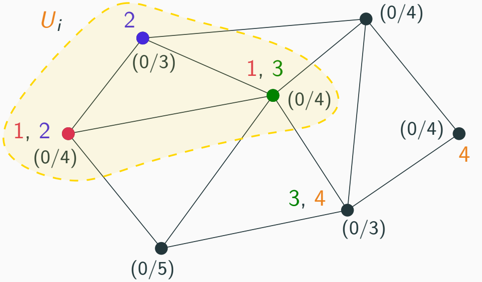

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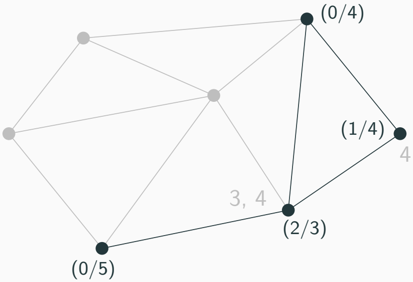

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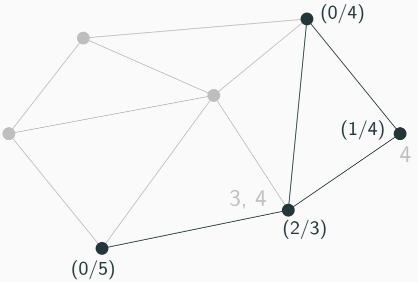

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Graph G is f-**DP-paintable** if Painter has a winning strategy on (G, f).

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Weak degeneracy vs online DP-coloring

Proposition

If G is weakly f-degenerate, then G is $(f+1)$-DP-paintable.

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Let G be weakly f-degenerate. Suppose that $g(u) + h(u) = f(u) - 1$ for each $u \in V(G)$. Then there is a partition $V(G) = V_1 \sqcup V_2$ such that $G[V_1]$ is weakly g-degenerate and $G[V_2]$ is weakly h-degenerate.

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Safe weak degeneration

Graph G is U-**safely weakly** f-**degenerate** for $U \subseteq V(G)$ if there is a sequence of legal Delete and DelSave operations where every vertex in U is removed using Delete.

Weak degeneracy of planar graphs

Lemma

Let G be a planar graph on at least 3 vertices, where every internal face is triangle, and the outer face is a cycle $C = (v_1, \dots, v_k)$. Define $f : V(G) \setminus \{v_1, v_2\} \to \mathbb{N}$:

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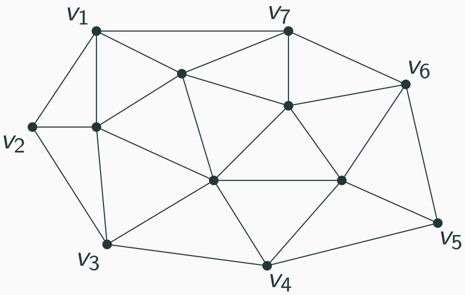

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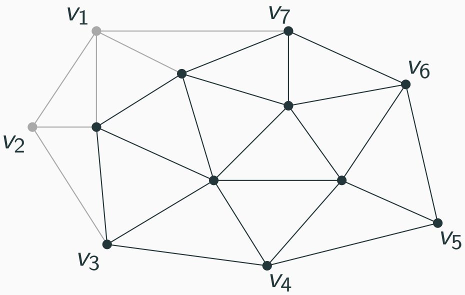

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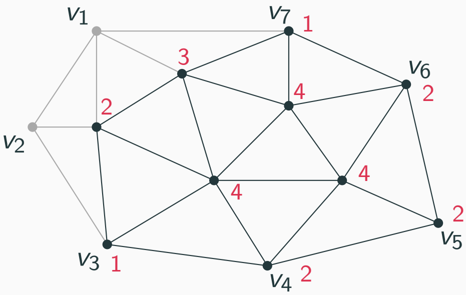

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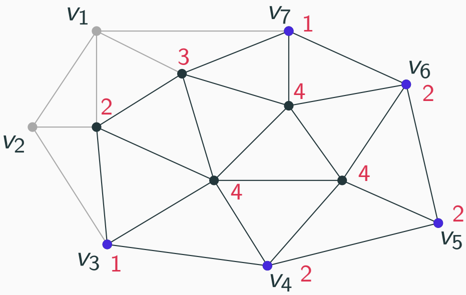

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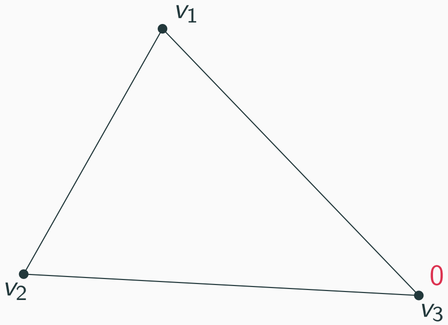

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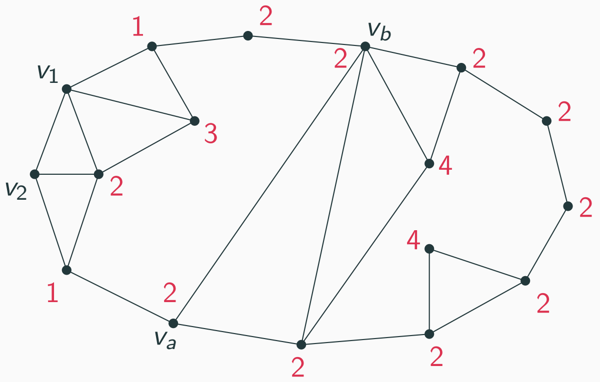

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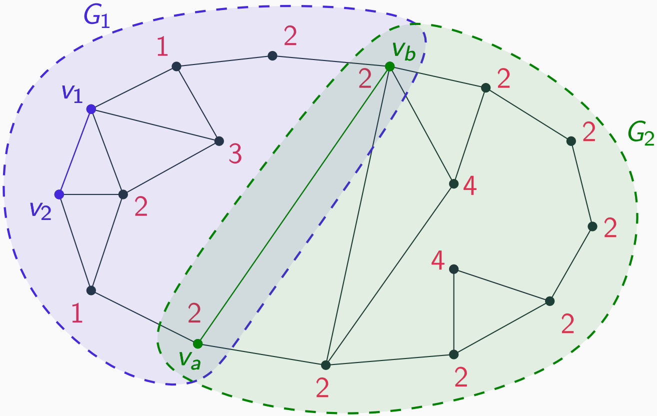

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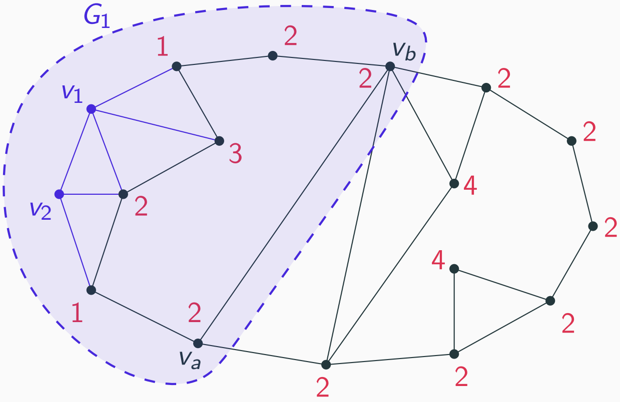

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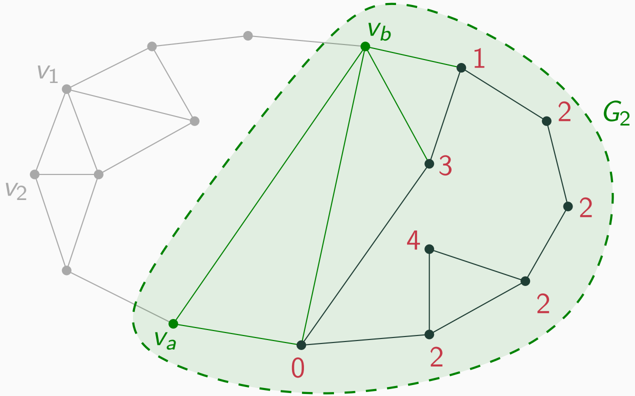

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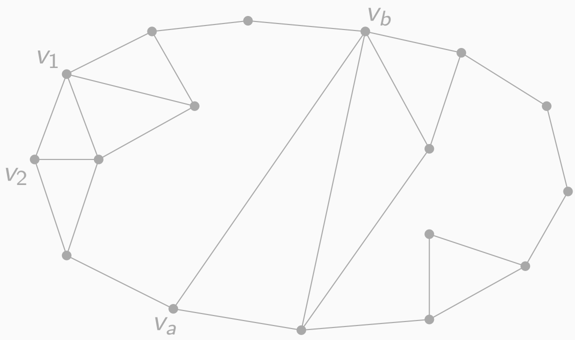

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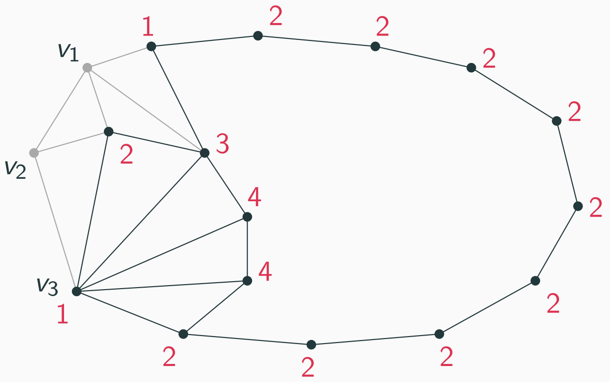

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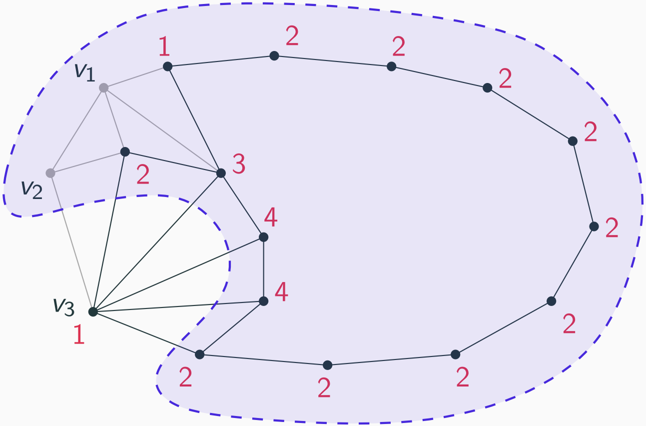

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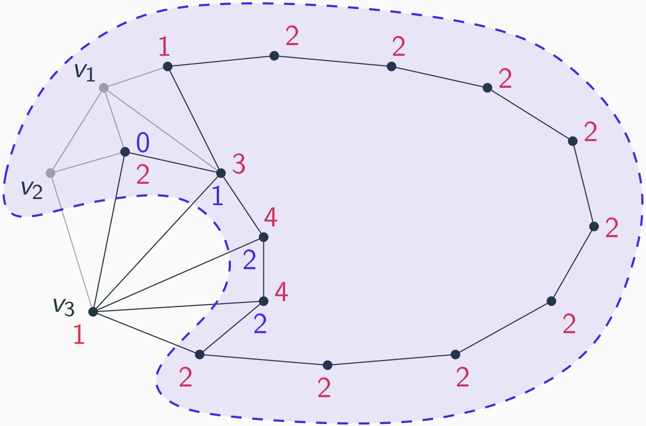

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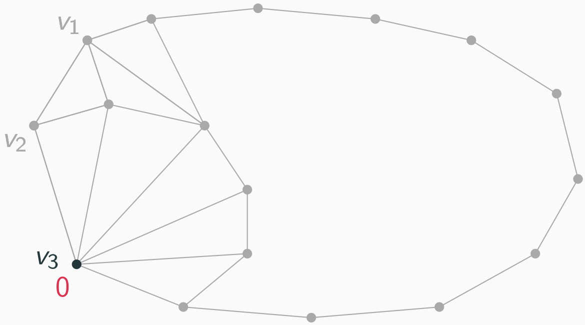

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If G is a connected graph with maximum degree $d \geq 3$, then either $G \cong K_{d+1}$ or G is weakly $(d-1)$-degenerate.

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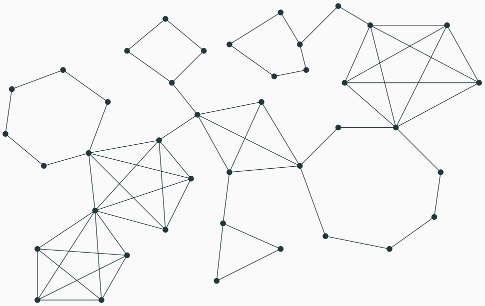
Graph G is **GDP-tree** if its each biconnected component is a cycle or clique.



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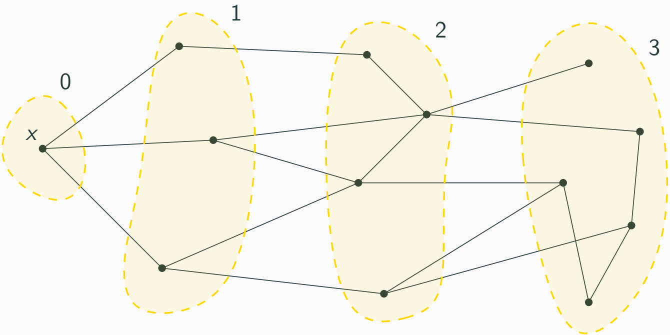
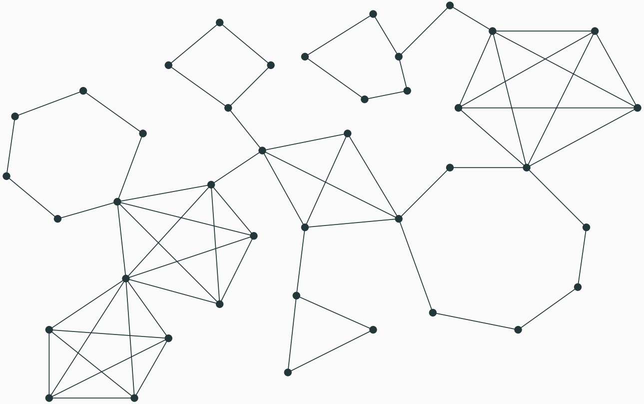
Remove vertices in decreasing distance order from x.



Lemma

Let G be a connected graph such that every biconnected induced subgraph of G is regular. Then G is GDP-tree.



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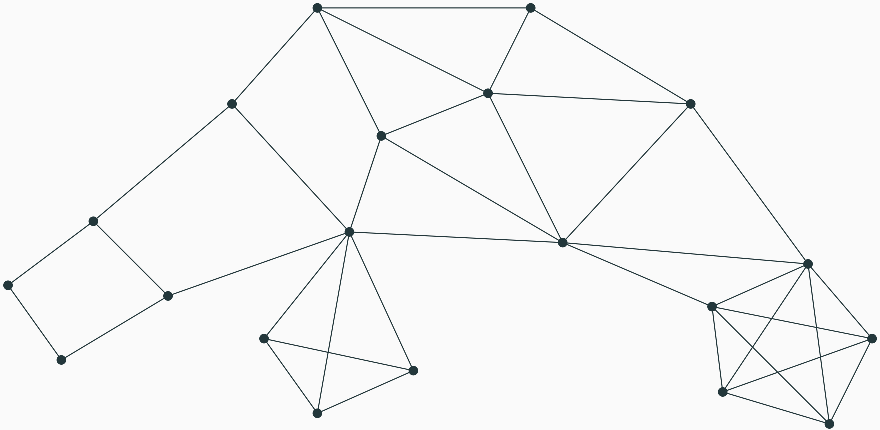

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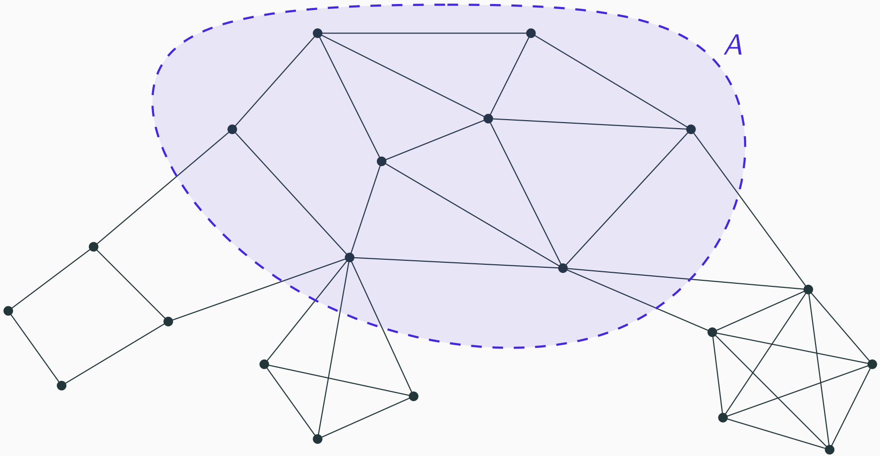

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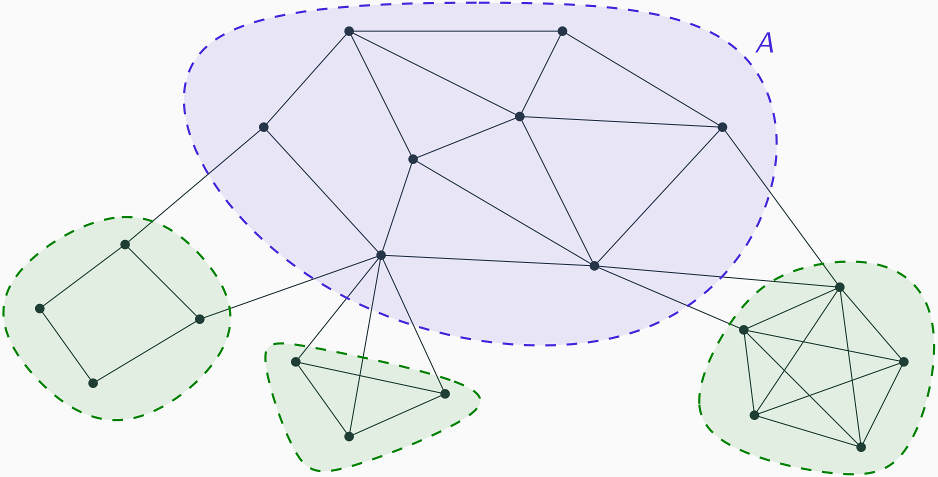

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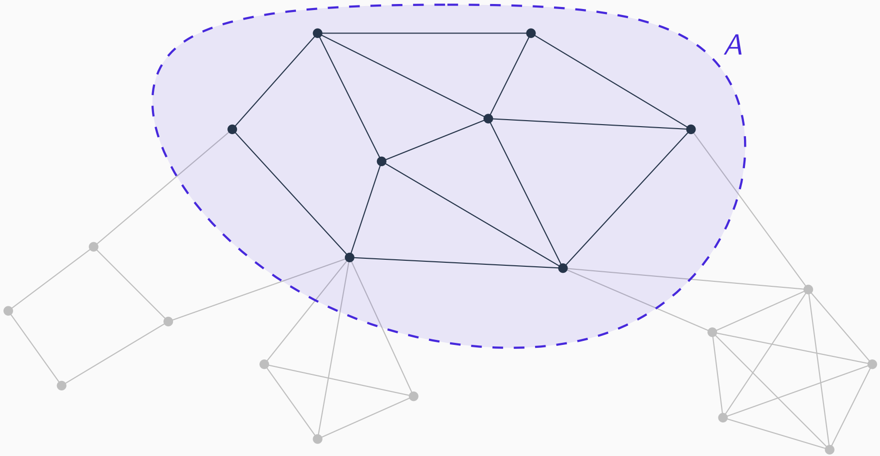

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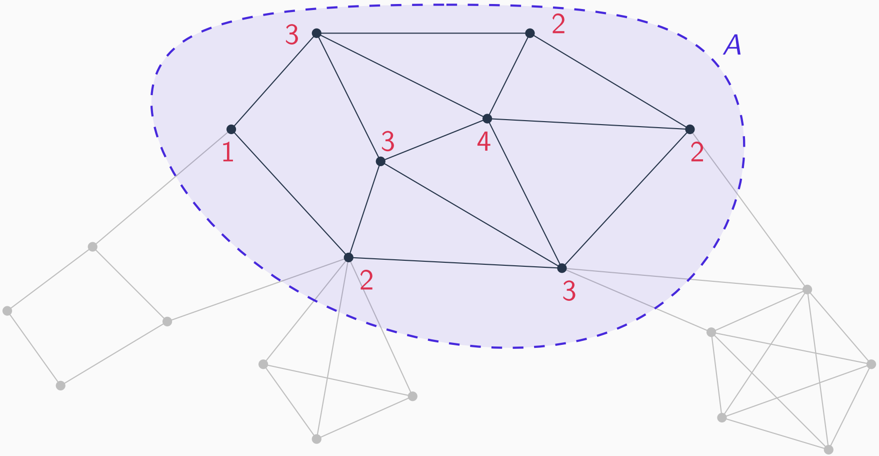

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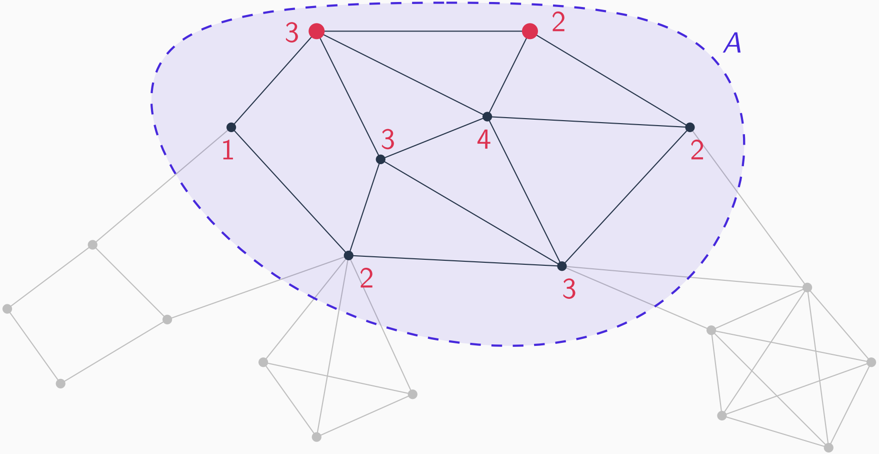

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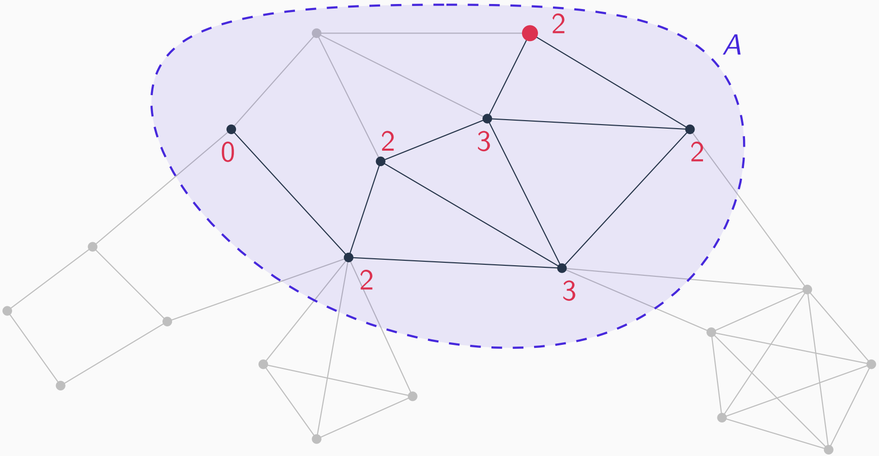

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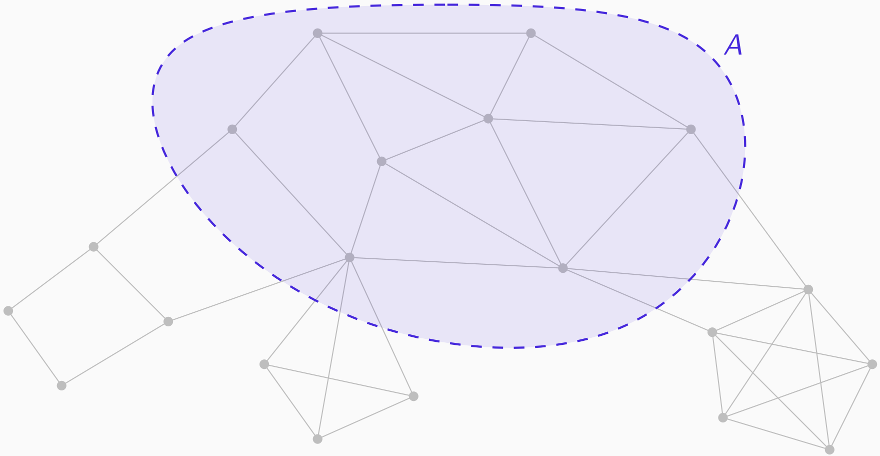

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$(1 \Rightarrow 2)$ GDP-trees are not DP-degree-colorable. [Bernshteyn, Kostochka, Pron 2017]

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Let G be a nonempty graph. If the weak degeneracy of G is at least $d \geq 3$, then either G contains a $(d+1)$-clique or

$$\operatorname{mad}(G) \geq d + \frac{d-2}{d^2 + 2d - 2}$$

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For each integer $k \geq 1$, there exists $c > 0$ and $d_0 \in \mathbb{N}$ such that if G is a graph of maximum degree $d \geq d_0$ with $\chi(G) \leq k$, then $\mathrm{wd}(G) \leq d - c\sqrt{d}$.

Theorem

There exists $c > 0$ and $d_0 \in \mathbb{N}$ such that if G is a graph of maximum degree $d \geq d_0$ and **girth at least 5**, then $\mathrm{wd}(G) \leq d - c\sqrt{d}$.

Conjecture

For each integer $k \geq 1$, there exists $c > 0$ and $d_0 \in \mathbb{N}$ such that if G is a graph of maximum degree $d \geq d_0$ and **without a k-clique**, then $\mathrm{wd}(G) \leq d - c\sqrt{d}$.

The authors don't know if the conjecture holds even for $k = 3$.