A Relative of Hadwiger's Conjecture

Demian Banakh
Department of Theoretical Computer Science
Jagiellonian University

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Overview

1. **Hadwiger's conjecture**
2. **Main result**
3. **Applications**

Notation

- We will consider only finite simple graphs (no loops or multiple edges).
- A graph H is a minor of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by edge-contraction.
- $\Delta(G)$ denotes the maximum degree of G.
- If $X \subseteq V(G)$, we denote by $G|X$ the subgraph of G induced on X.
- $X \subseteq V(G)$ is called an independent set of G if $\Delta(G|X) = 0$.

Hadwiger's conjecture

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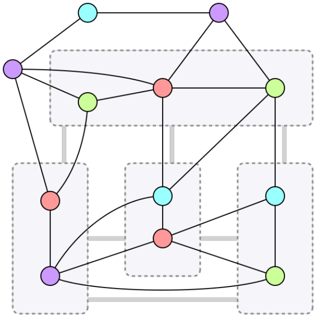

Source: Wikipedia

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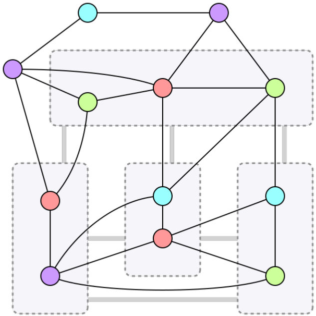

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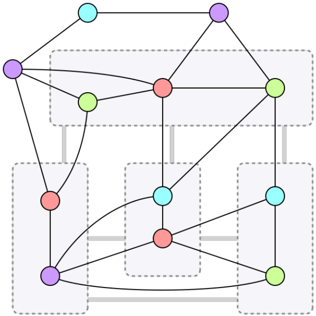

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- This is still an open question, even though the conjecture has been proven for $t \leq 5$.
- In the 1980s it was proven that every graph with no K_t minor has average degree $O(t\sqrt{\log t})$ (the more formal statement of which we will see later) and hence is $O(t\sqrt{\log t})$-colorable. In [Postle, 2020], this bound was improved to $O(t \cdot (\log \log t)^6)$.

Hadwiger's conjecture

To match the statement of the main result presented today, let's rephrase the conclusion of Hadwiger's conjecture in terms of vertices partition.

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For all integers $t \geq 0$, and every graph G, if K_{t+1} is not a minor of G, then $V(G)$ can be partitioned into t independent sets, i.e. sets $X_1, \ldots, X_t$ such that $\Delta(G|X_i) = 0$ for $1 \leq i \leq t$.

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Note

It is a strong bound for the size of a partition - result becomes false if we ask for a partition into $t - 1$ independent sets.

Main result

Theorem

For all integers $t \geq 0$ there is an integer s such that for every graph G, if K_{t+1} is not a minor of G, then $V(G)$ can be partitioned into t sets $X_1, \ldots, X_t$ such that $\Delta(G|X_i) \leq s$ for $1 \leq i \leq t$.

Note

Such partitions are often called *defective colorings* in the literature.

Main result: strong bound

Despite being much weaker than Hadwiger's conjecture, it still exhibits a strong bound for the size of a partition in the same sense.

Theorem

For all integers $s \geq 0$ and $t \geq 1$, there is graph $G = G(s,t)$ such that K_{t+1} is not a minor of G, and there is no partition $X_1, \ldots, X_{t-1}$ of $V(G)$ into $t-1$ sets such that $\Delta(G|X_i) \leq s$ for $1 \leq i \leq t-1$.

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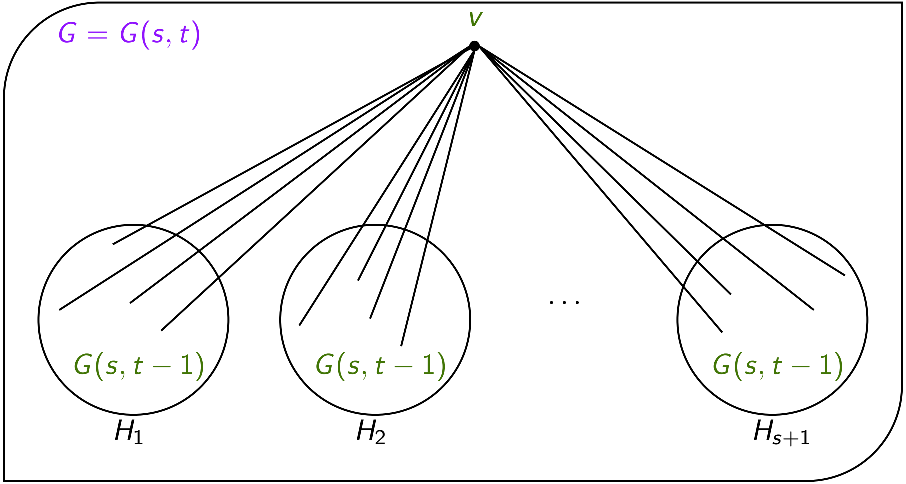
Proof

The proof is by induction on t. Construct $G = G(s,t)$ as follows. Note that G has no K_{t+1} minor, since each H_i has no K_t minor.

Main result: strong bound



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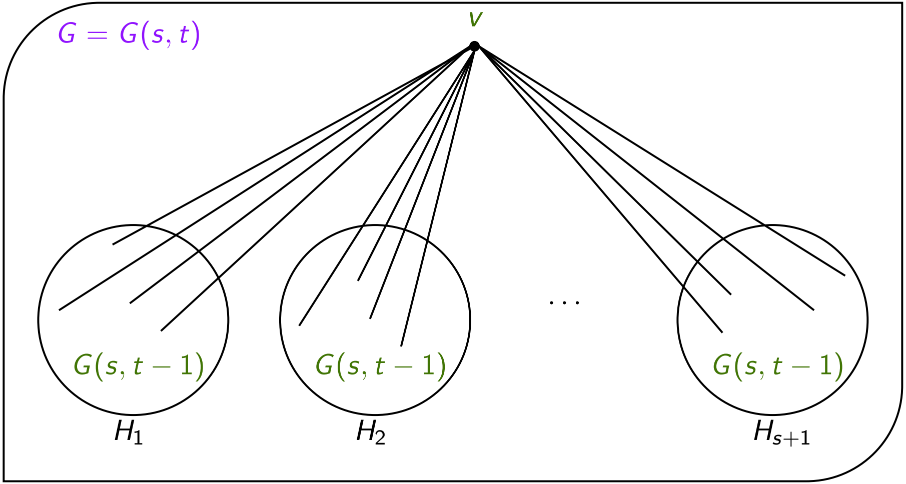

For the sake of contradiction, assume that such partition $X_1, \dots, X_{t-1}$ of $V(G)$ exists ($\forall i\ \Delta(G|X_i) \leq s$). Without loss of generality, let $v \in X_{t-1}$.

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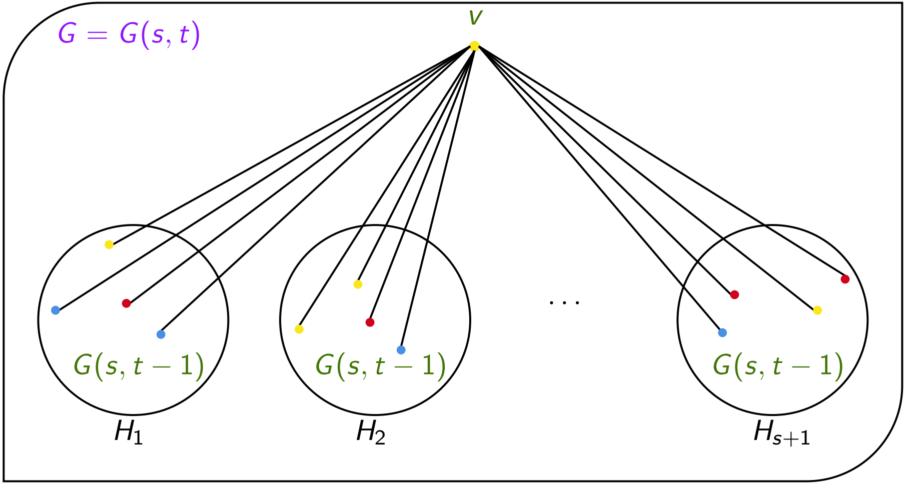

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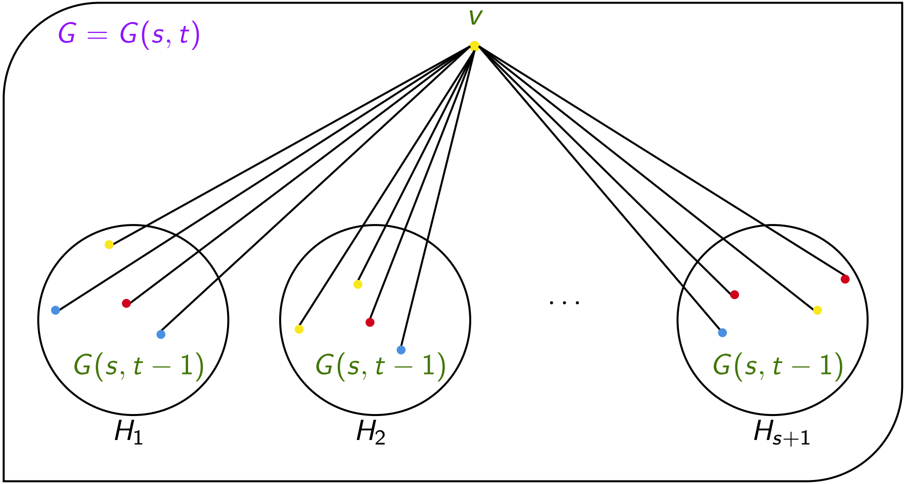

One of the copies H_i must have no elements in common with X_{t-1}. Without loss of generality $X_{t-1} \cap V(H_1) = \emptyset$.

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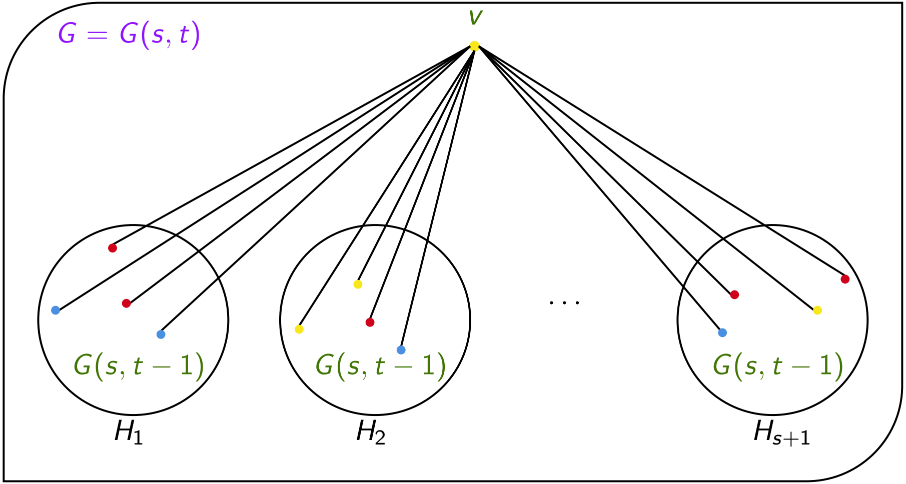

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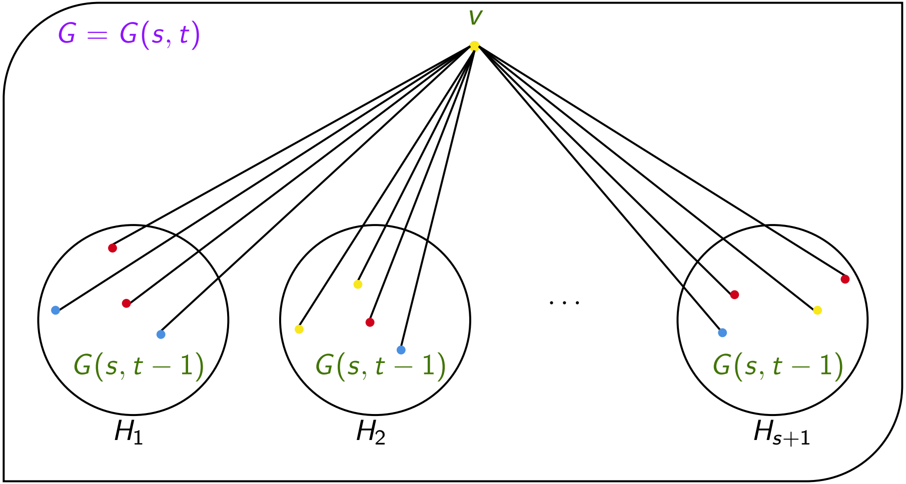

The other sets of partition restricted to $V(H_1)$ turn out to be a partition of $G(s, t-1)$ with maximum degree at most s, which is a contradiction.

Main result: strong bound

More formally, let $Y_i = X_i \cap V(H_1)$ for $1 \leq i \leq t-2$. Then $Y_1, \ldots, Y_{t-2}$ provide a partition of $V(H_1)$ into $t-2$ sets, and since H_1 is isomorphic to $G(s, t-1)$, it follows that $\Delta(H_1|Y_i) > s$ for some $1 \leq i \leq t-2$, a contradiction to $\Delta(G|X_i) \leq s$.

Proof of Main Result

Theorem

Let $t \geq 0$ be an integer, and let s be as in Lemma (3). For every graph G, if K_{t+1} is not a minor of G, then $V(G)$ can be partitioned into t sets $X_1, \ldots, X_t$ such that $\Delta(G|X_i) < s$ for $1 \leq i \leq t$.

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We proceed by induction on $|V(G)| + |E(G)|$.

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We proceed by induction on $|V(G)| + |E(G)|$.

- If some vertex v of G has $\Delta(v) < t$, the result follows from the inductive hypothesis by deleting v (find a partition by induction and add v to some set X_i that contains no neighbor of v).

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- If some vertex v of G has $\Delta(v) < t$, the result follows from the inductive hypothesis by deleting v (find a partition by induction and add v to some set X_i that contains no neighbor of v).
- If some edge e has both ends of $\Delta < s$, the result follows from the inductive hypothesis by deleting e (find a partition by induction and note that inserting e back will not cause either of the ends of e to have degree too large).

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The rest of this section is dedicated to arguing that at least one of these cases must hold.

Proof of Main Result

Lemma (1)

There exists $C > 0$ *such that for all integers* $t \geq 0$ *and all graphs* G*, if* K_{t+1} *is not a minor of* G*, then* G *has at most* $C(t+1)(\log(t+1))^{\frac{1}{2}} \cdot |V(G)|$ *edges.*

Proof of Main Result

Lemma (2)

Let $t \geq 0$ *be an integer, let* C *be as in Lemma (1), and let* $r \geq C(t+1)(\log(t+1))^{\frac{1}{2}}$. *Let* G *be a graph such that* K_{t+1} *is not a minor of* G*, and let* $A \subseteq V(G)$ *be an independent set of vertices each of degree at least* t. *Then*

$$|E(G \setminus A)| + |A| \leq r|V(G \setminus A)|$$

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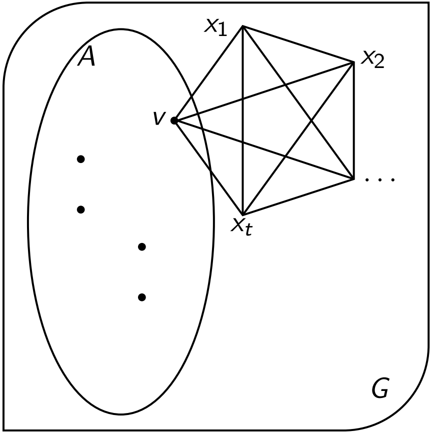
Proof

The proof is by induction on $|A|$. If $A = \emptyset$, we refer to Lemma (1) directly.

Proof of Lemma (2)



- Otherwise, let $v \in A$, then $\Delta(v) \geq t$.

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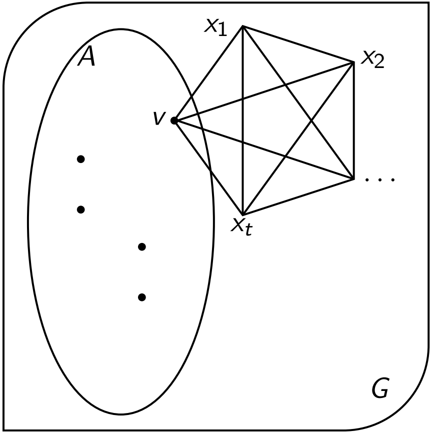

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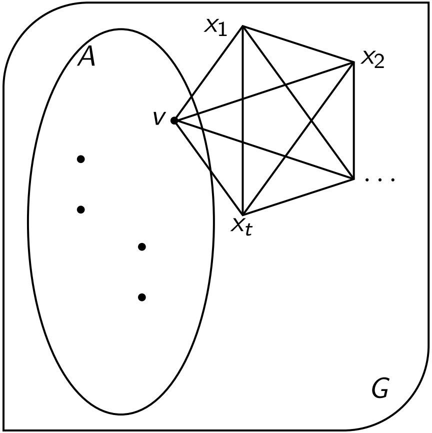

- Otherwise, let $v \in A$, then $\Delta(v) \geq t$.
- If every two neighbors of v were adjacent, we would get that K_{t+1} is a subgraph of G.
- So v has two neighbors x, y which are non-adjacent to each other.

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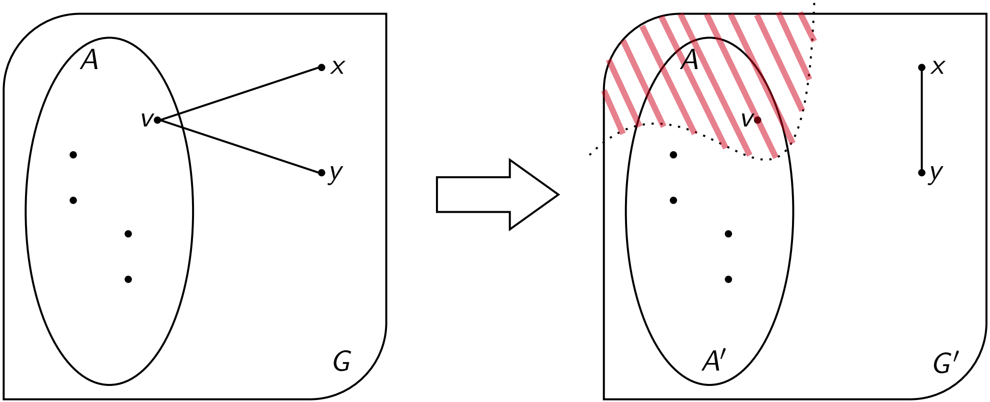

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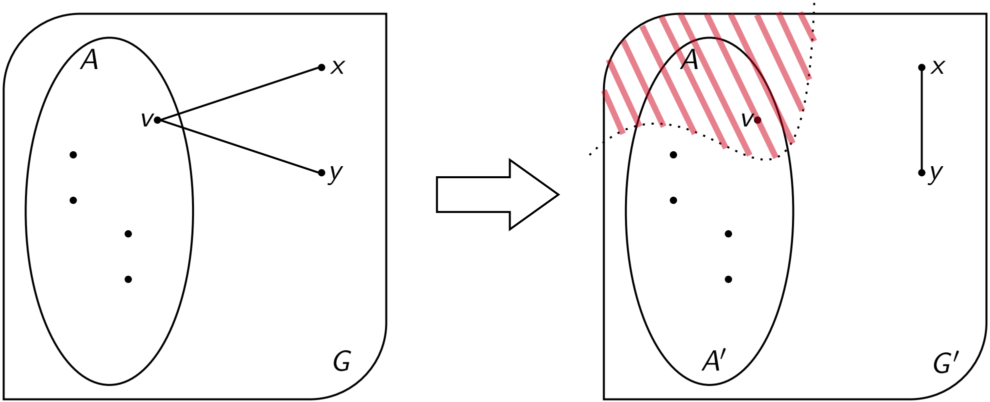

Formally, let $G' = (G \setminus v) + xy$ and $A' = A \setminus \{v\}$. Note that $|V(G' \setminus A')| = |V(G \setminus A)|$, $|E(G' \setminus A')| = |E(G \setminus A)| + 1$ and $|A'| = |A| - 1$.

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K_{t+1} is not a minor of G', since G' is a minor of G. It follows from the inductive hypothesis that $|E(G' \setminus A')| + |A'| \leq r|V(G' \setminus A')|$. It is enough to apply the aforementioned equalities now.

Proof of Main Result

Lemma (3)

Let $t \geq 0$ be an integer, let C be as in Lemma (1), and let $r \geq C(t+1)(\log(t+1))^{\frac{1}{2}}$ and $r > \frac{t}{2}$. Let $s > r(2r - t + 2)$. Let G be a nonnull graph such that K_{t+1} is not a minor of G. Then either

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- *there are 2 adjacent vertices both with degree less than s.*

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Assume that $t \geq 2$, for if $t \leq 1$ the result is trivially true.
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We may also assume that no two vertices of A are adjacent because otherwise the second outcome holds.

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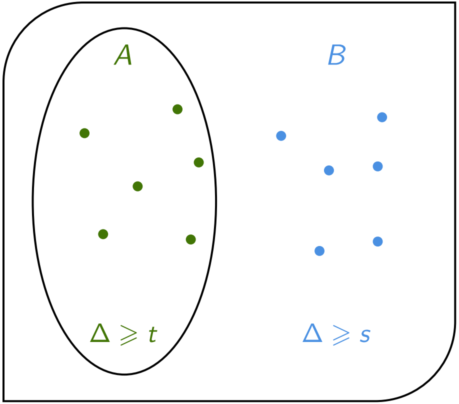

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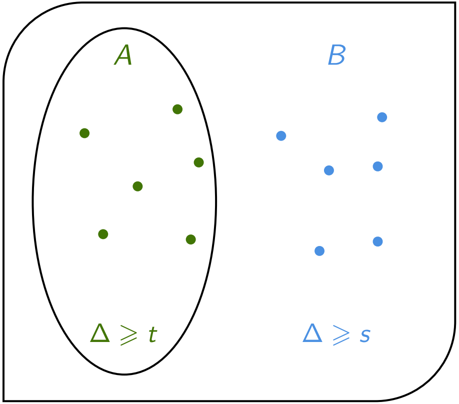

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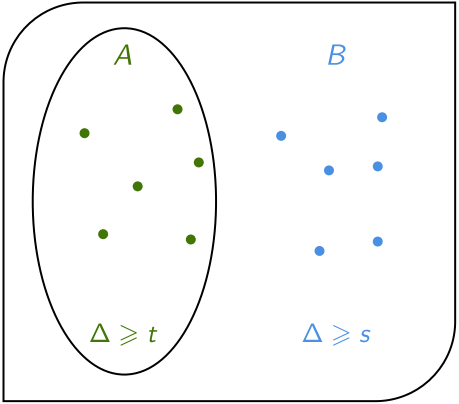

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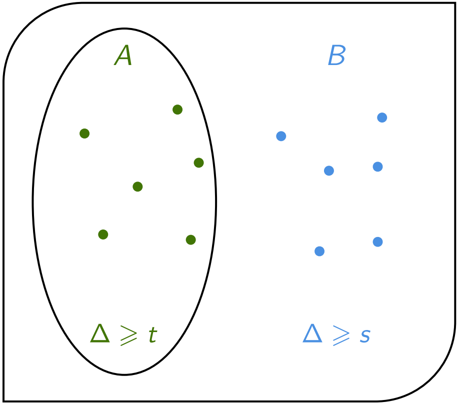

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- It follows that

$$t|A| + s|B| \leq 2r(|A| + |B|),$$

that is,

$$|A| \geq \frac{s - 2r}{2r - t}|B|,$$

since $2r > t$.

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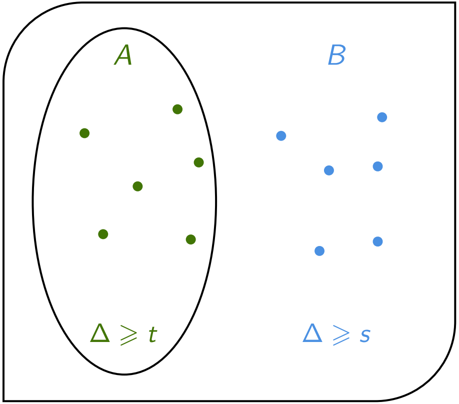

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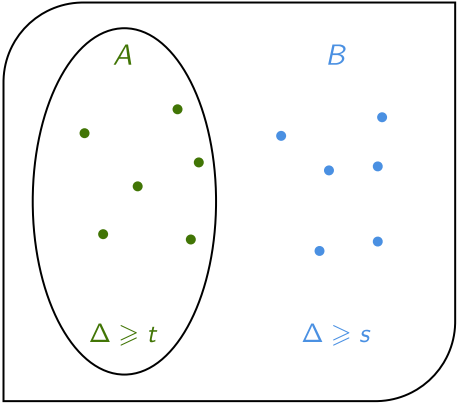

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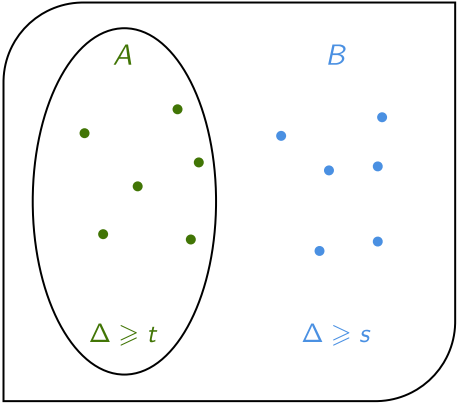

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- By Lemma (2),
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- Since G is a nonnull graph, $|B| \neq 0$, and so
$$r \geq \frac{s-2r}{2r-t},$$
that is,
$$s \leq r(2r-t+2),$$
a contradiction.

Partitions into sets inducing graphs with no large component

In [Kawarabayashi, Mohar, 2007], the following variant of defective colorings was proven.

Theorem

There is a function $f(t) \in O(t)$ *and a computable function* $s(t)$ *such that if* G *is a graph with no* K_{t+1} *minor, then* $V(G)$ *can be partitioned into* $f(t)$ *sets, inducing subgraphs in which every component is size at most* $s(t)$.

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In particular, the authors showed that taking $f(t) = \lceil 15.5(t+1) \rceil$ works. Later this was improved to $f(t) = 3t$ in [Liu, Oum, 2018]. That suggests a nice open question - can we prove the same with $f(t) = t$?

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For all integers $t, \Delta \geq 0$, there exists s such that for every graph G, if K_{t+1} is not a minor of G and $\Delta(G) \leq \Delta$, then $V(G)$ can be partitioned into four sets X_1, X_2, X_3, X_4 such that every component of $G|X_i$ has at most s vertices.

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For a graph G with no K_{t+1} minor, we first partition it into t sets each inducing a subgraph of bounded degree. Finally, we apply the theorem above to every set in the partition to obtain a partition of $V(G)$ into $4t$ sets each inducing a graph with no large component.

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The End