Recap

List coloring - Given a graph G and a set L(v) of colors for each vertex v, a list coloring is a choice function that maps every vertex v to a color in the list L(v), and no two adjacent vertices receive the same color.

k-choosability - A graph is k-choosable if it has a proper list coloring no matter how one assigns a list of k colors to each vertex.

Minor - A graph H is called a minor of the graph G if H can be formed from G by deleting edges and vertices and by contracting edges.

Wagner's theorem - A finite graph is planar if and only if it does not have K_5 or $K_{3,3}$ as a minor.

Theorems

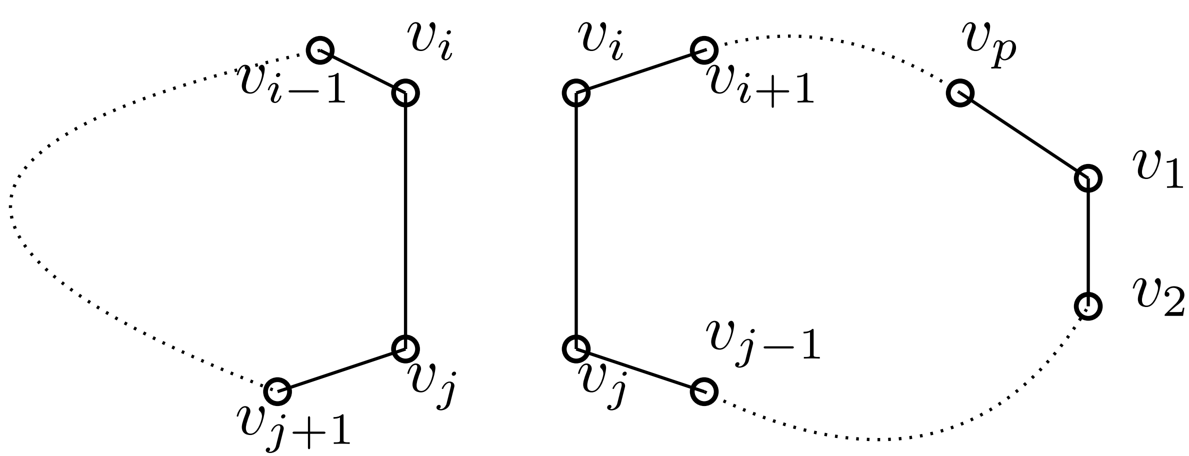
Theorem 1 (Thomassen, 1994) All planar graphs are 5-choosable.

Theorem 2 (Skrekovski, 1998) All K_5-minor free graphs are 5-choosable.

Proof of the theorem 1

Lemma 1. Let G be a near-triangulation with an outer cycle C : $v_1 v_2 \dots v_p$ and L be a list assignment of G such that $|L(v)| \geq 3$ for $v \in C$ and $|L(v)| \geq 5$ for $v \in V \setminus C$. Suppose that λ is a coloring of $\{v_1, v_2\}$. Then λ can be extended to a coloring of G

Case 1. Outer cycle has a chord - $v_j v_i$.

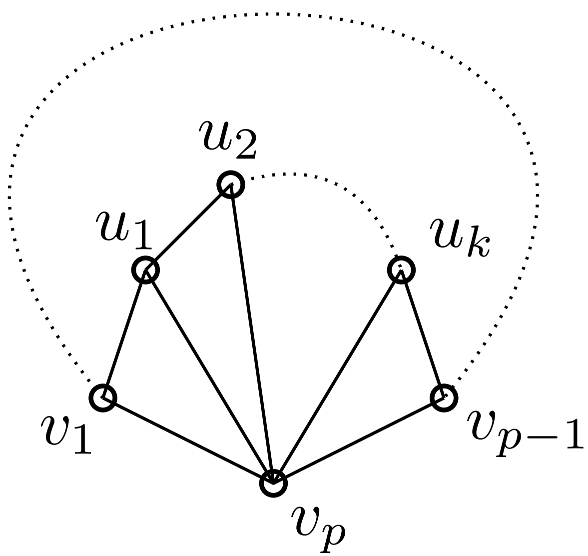


WLOG $v1$ and v_2 are in the right part.
We can apply the induction hypothesis to color the right part.
This fixes colors for v_i and v_j, so we can apply the induction hypothesis to color the left part.

Proof of the theorem 1

Case 2. Outer cycle doesn't have a chord.

Let $v_1, u_1, u_2, \ldots, u_k, v_{p-1}$ be the neighbors of v_p, in that order.



Let $c_1, c_2 \in L(v_p)$ be colors different from the one fixed for v_1.

Let's remove c_1, c_2 from $L(u_i)$ for all i.

Let's apply the induction hypothesis to the graph without the v_p.

We can color v_p with either c_1 or c_2 depending on which color is assigned to the vertex v_{p-1}.

Lemma 2. . Let G be a near-triangulation and L be a list assignment of G such that $|L(v)| \geq 5$ for every $v \in V(G)$. Suppose that H is a subgraph of G isomorphic to K_3 or K_2 and λ is a coloring of H. Then λ can be extended to a coloring of G.

Case 1. $H \cong K_2$

We may assume that H lies on the outer cycle of G and G is near-triangulation. In that case we can use lemma 1 to extend the λ.

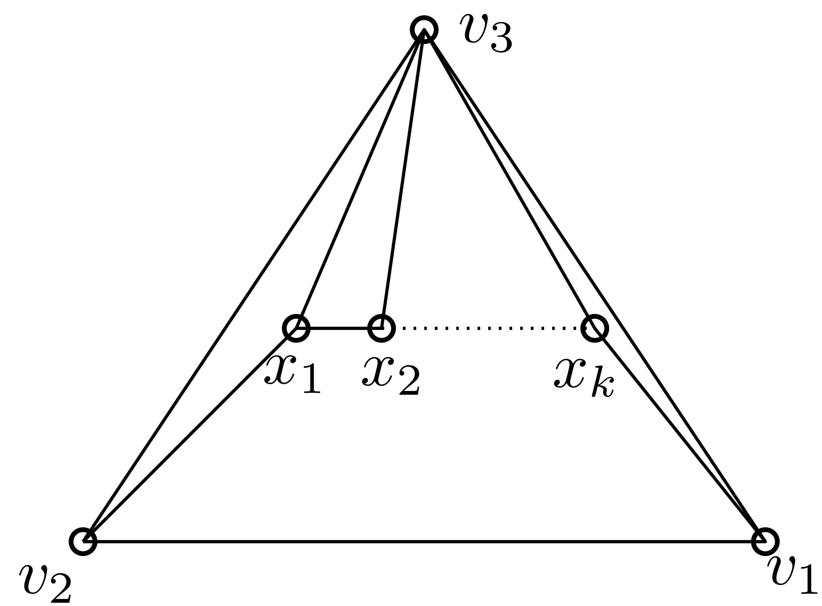
Case 2. H is not a separating cycle.

We may assume that H is an outer face and that G is a near-triangulation.

Let $v_2, x_1, x_2, \ldots, x_k, v_1$ be the neighbors of v_3, in that order.

Let's remove $\lambda(v_3)$ from $L(x_i)$'s.

We can use lemma 1 to color $G \setminus \{v_3\}$.

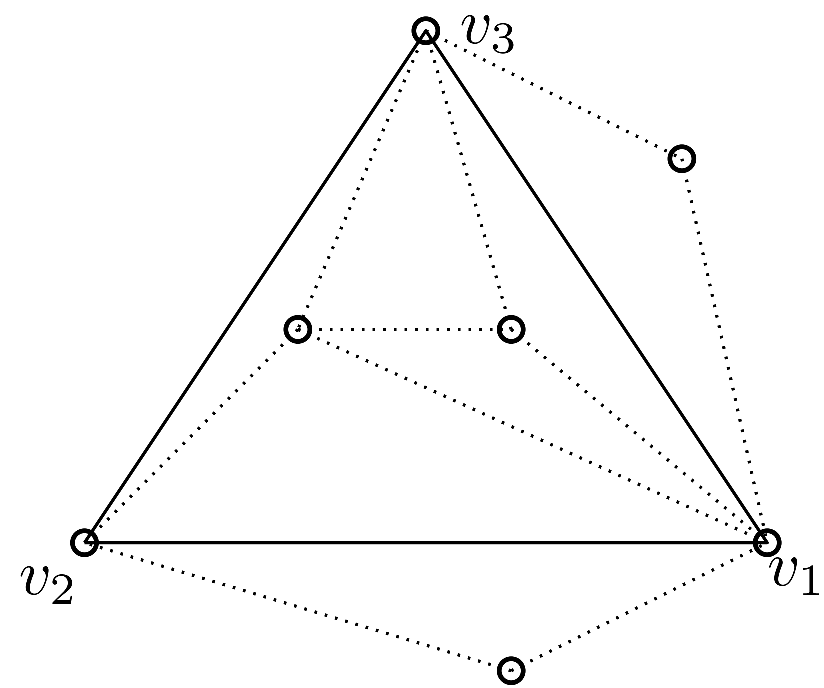


Case 3. H is a separating cycle.

Let H_1 be the outer part, and H_2 be the inner part.

Let $G_1 = H_1 \cup H$ and $G_2 = H_2 \cup H$.

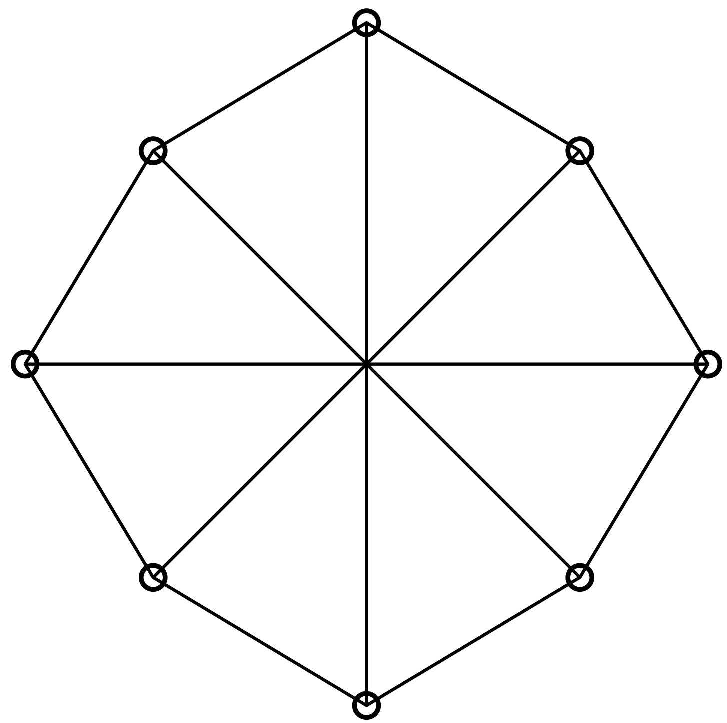
We can apply the same logic as in case 2 to both G_1 and G_2.



K_5-minor-free graphs characterization

Clique-sum is a way of combining two graphs by gluing them together at a clique.

Wagner graph (V_8) - the graph obtained from a cycle of length 8 by connecting opposite nodes.



Theorem 3. (Wagner) A graph G has no K_5 minor if and only if it can be obtained by 0-, 1-, 2- and 3-clique-sum operations from planar graphs and V_8.

Theorem 2, Proof 1

Lemma 3. Let G be an edge-maximal K_5-minor-free graph and let L be a list assignment of G such that $|L(v)| \geq 5$ for every vertex $v \in V(G)$. Suppose that H is a subgraph of G isomorphic to K_2 or K_3, and λ is a coloring of H. Then λ can be extended to a coloring of G.

Case 1. G is planar

It follows from lemma 2.

Case 2. $G \cong V_8$

Degree of every vertex is 3, so λ can be greedily extended to G.

Case 3.

From theorem 3 it follows that $G = G_1 \cup G_2$ where G_1, G_2 are proper subgraphs of G such that $G_1 \cap G_2 = K_2$ or K_3.

WLOG $H \subseteq G1$. By the induction hypothesis applied to G_1, λ can be extended to a coloring of G_1.

By the induction hypothesis applied to G_2 with $H' = G_1 \cap G_2$, λ can be extended to a coloring of G_2.

Theorem 2, Proof 2

Another proof of lemma 3 from the Skrekovski's paper.

Lemma 4 (Halin). Every 4-connected non-planar graph contains K_5 as a minor.

Lemma 5. Let G be a 3-connected non-planar graph with only one 3-cut T. Suppose that $G \setminus T$ has exactly two components. Then G contains K_5 as a minor.

Proof sketch for both: Assume that G contains $K_{3,3}$ as a minor. Use 4-connectivity or unique 3-cut to get the K_5 minor from the $K_{3,3}$ minor.

Lemma 3. Let G be an edge-maximal K_5-minor-free graph and let L be a list assignment of G such that $|L(v)| \geq 5$ for every vertex $v \in V(G)$. Suppose that H is a subgraph of G isomorphic to K_2 or K_3, and λ is a coloring of H. Then λ can be extended to a coloring of G.

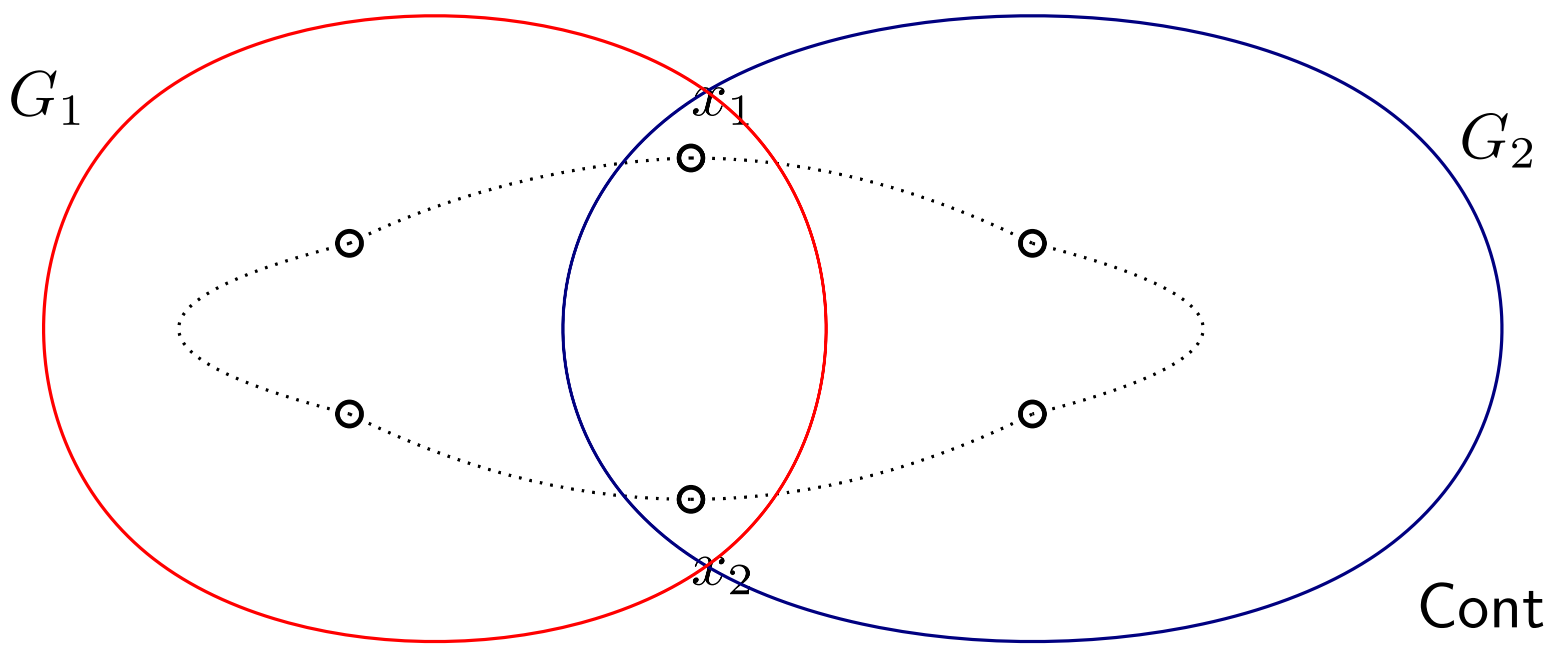
Theorem 2, Proof 2

Proof by contradiction: Let the G be a counterexample with minimal $|V|$.

Let $T = \{x_1, x_2, \ldots, x_t\}$ be a minimal cut of G. Let G_1, G_2 be subgraphs of G such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = T$. WLOG $H \subseteq G_1$.

Case 1. G is not 3 connected.

Case $t = 1$ is trivial. Let $t = 2$. Let $H_1 = G_1 \cup \{x_1 x_2\}$ and $H_2 = G_2 \cup \{x_1 x_2\}$. Both H_1, H_2 are K_5-minor-free. We can apply "induction" to color the H_1 and then do the same to H_2 with $H' = T$.



Contradiction.

Theorem 2, Proof 2

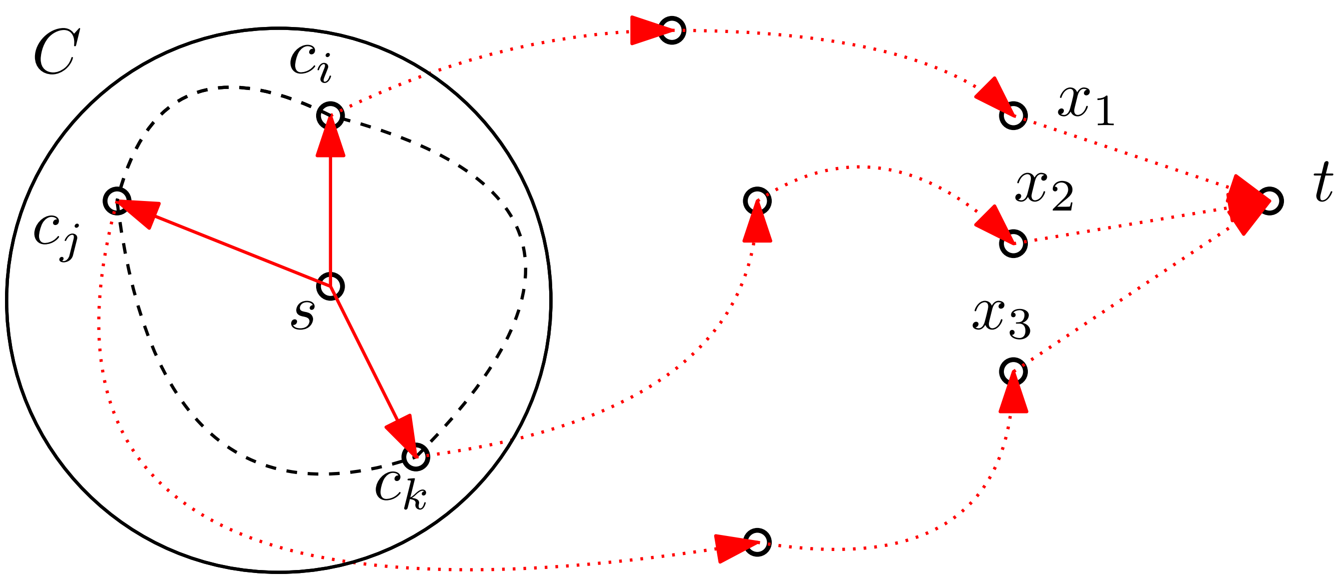
Case 2. G is planar.

Contradiciton follows from lemma 2.

Case 3. $G_2 \ncong K_{3,1}$ and $G_1 \ncong K_{3,1}$

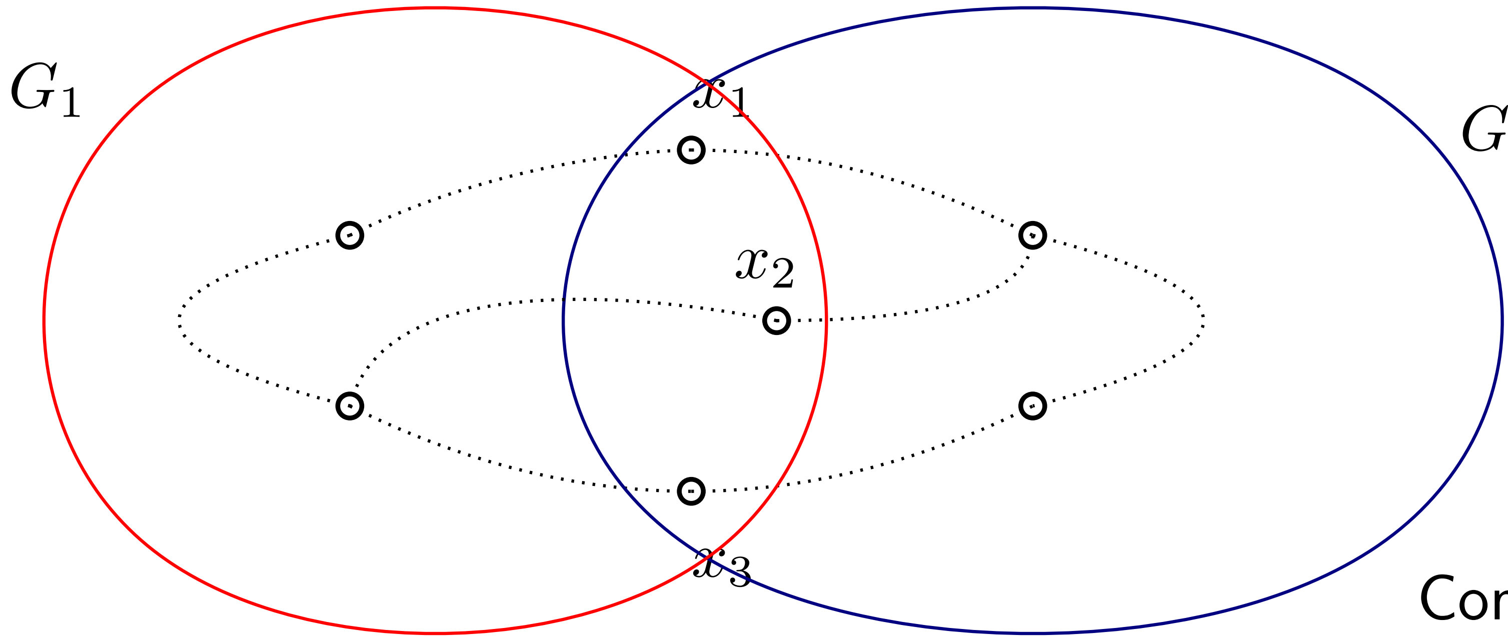
Claim 1. G_i can be contracted to K_3 whose vertecies are $\{x_1, x_2, x_3\}$.

G_i contains a cycyle C. From max-flow min-cut it follows that there exist 3 vertex-disjoint paths from C to x_1, x_2, x_3. We can contract those paths, and then contract the cycle to K_3.



Theorem 2, Proof 2

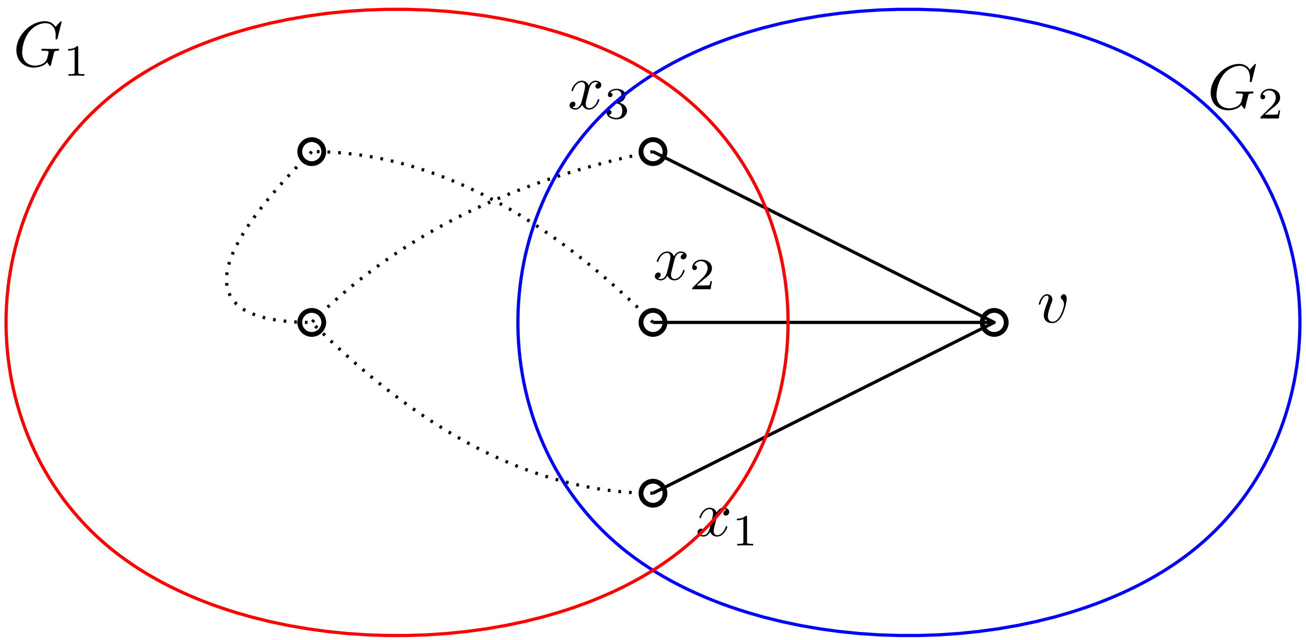
Let $H_1 = G_1 \cup \{x_1x_2, x_1x_3, x_2x_3\}$ and $H_2 = G_2 \cup \{x_1x_2, x_1x_3, x_2x_3\}$. From claim 1 it follows that both H_1, H_2 are K_5-minor-free. We can apply "induction" to color the H_1 and then do the same to H_2 with $H' = T$.



Contradiction.

Theorem 2, Proof 2

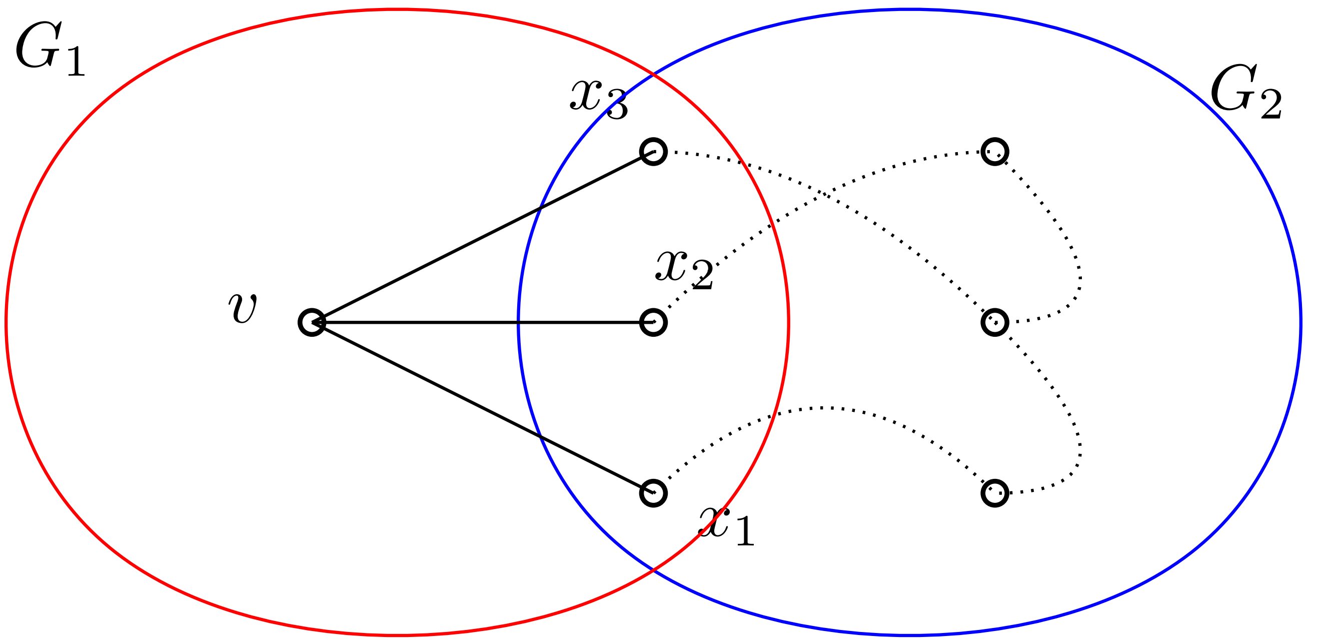
Case 4. $G_2 \cong K_{3,1}$.



Let's remove v and then apply the "induction" to get the coloring for G_1.
Then we can color v with $c \in L(v) \setminus \{\lambda(x_1), \lambda(x_2), \lambda(x_3)\}$.
Contradiction.

Theorem 2, Proof 2

Case 5. $G_1 \cong K_{3,1}$.



We can assume that $G_2 \setminus T$ has only one connected component because otherwise we could "move" one of the G_2's component to G_1.
Let's assume that there exists another 3-cut T'. Then if we take T' instead of T then $G_1 \ncong K_{3,1}$. Contradiction.
Combining those two properties we have that T is the only 3-cut of G and $G \setminus T$ has only 2 connected components. That contradicts lemma 5.