Graph Coloring Game

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Theoretical Computer Science

Definition

Graph coloring game

In the *graph coloring game* two players Alice and Bob are given graph **G** and a set of **k** colors. Alice and Bob take turns, **coloring properly** an uncolored vertex.

- Alice wins when graph is completely colored
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Game chromatic number

The game chromatic number of a graph **G**, denoted by $\chi_g(\mathbf{G})$, is the minimum number of colors needed for Alice to win the graph coloring game on **G**.

Example

Trees

Upper bound [Faigle, Kern, Kierstead, Trotter 1993]

If $\mathbf{G}$ is a tree, then $\chi_g(\mathbf{G}) \leq 4$.

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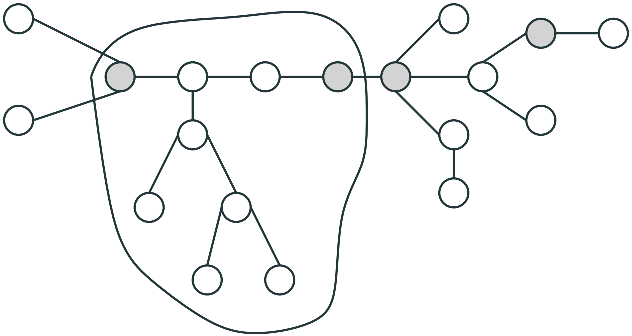

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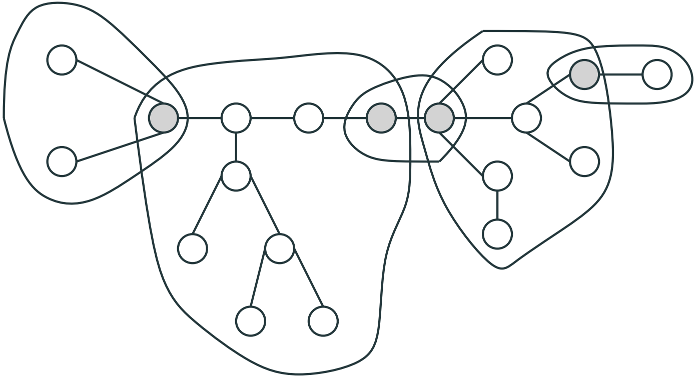

Trees

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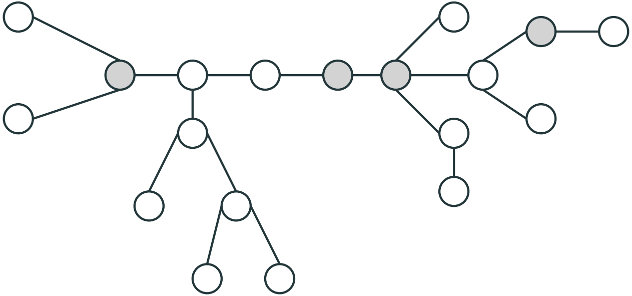
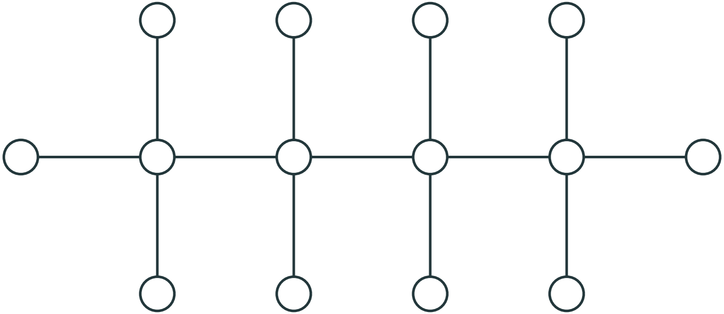
If G is a tree, then $\chi_g(G) \leq 4$.



Trees

Lower bound [Bodlaender 1991]

There exists a tree **T**, such that $\chi_g(\mathbf{T}) = 4$



Game coloring number

Marking game

In the *marking game* two players Alice and Bob are given graph **G**. During the game they create linear order L of vertices of graph **G**. They alternate turns with Alice playing first. In each move player selects a vertex from the reamining vertices and puts it at the end of L.

- Alice's goal is to **minimize** back degree of L
- Bob's goal is to **maximize** back degree of L

Game coloring number

Marking game

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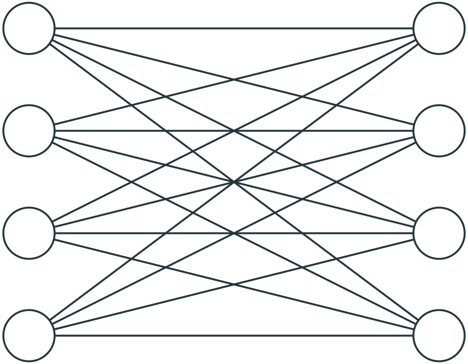
Game coloring number

The game coloring number of a graph $\mathbf{G}$, denoted by $\mathbf{col_g}(\mathbf{G})$, is equal to $k+1$, where k is back degree of a linear order L, which is produced by playing the **marking game** with both players using their **optimal** strategies.

Game coloring number



$\mathbf{col_g}(\mathbf{G}) = n + 1$

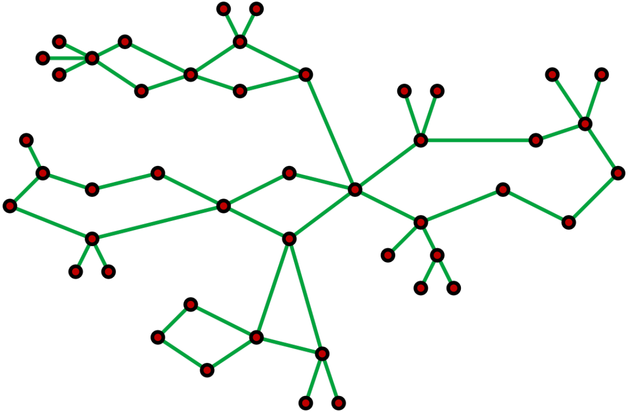
$\chi_g(\mathbf{G}) = 3$

Helpful lemma

Lemma

Suppose $\mathbf{G = (V\,,E)}$ and $\mathbf{E} = \mathbf{E_1} \cup \mathbf{E_2}$. Let $\mathbf{G_1} = (\mathbf{V}, \mathbf{E_1})$ and $\mathbf{G_2} = (\mathbf{V}, \mathbf{E_2})$.
Then $\mathbf{col_g}(\mathbf{G}) \leq \mathbf{col_g}(\mathbf{G_1}) + \Delta(\mathbf{G_2})$.

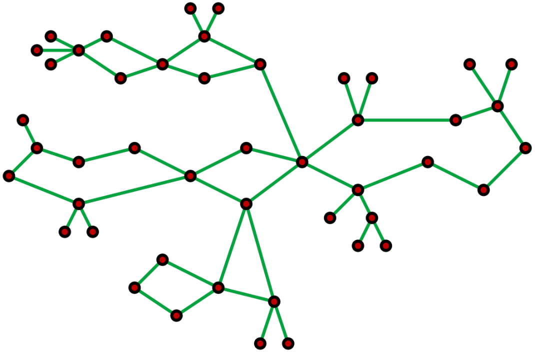
Cactuses



Cactuses

Lemma

If $\mathbf{G}$ is a cactus then there is a matching $\mathbf{M}$ such that $\mathbf{K} - \mathbf{M}$ is an acyclic graph.



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Thus edges of cactus $\mathbf{G}$ can be splitted into two graphs $\mathbf{G_1}, \mathbf{G_2}$,
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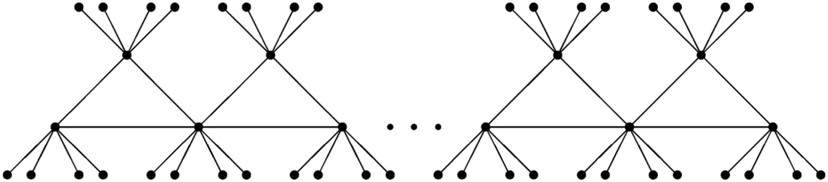
Upper bound [Sidorowicz 2006]

If G is a cactus, then $\chi_g(G) \leq 5$.

Cactuses

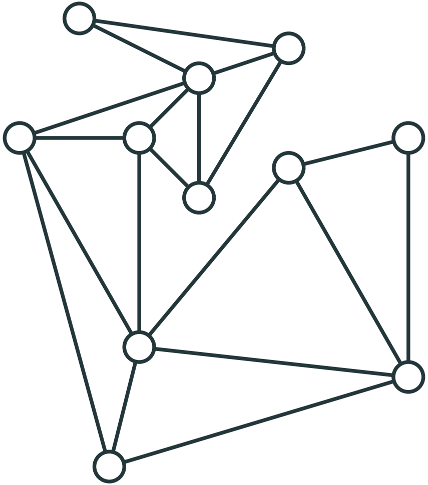
Lower bound [Sidorowicz 2006]

There exists a cactus **K**, such that $\chi_g(\mathbf{K}) = 5$



G contains 7 triangles

Outerplanar graphs



Outerplanar graphs

Theorem

For every maximal outerplanar graph there exists ordering of vertices $\{v_1, v_2, \ldots v_n\}$ such that:

- $v_1 v_2$ is an edge on the outer face,
- $\forall_{i>2}\ v_i$ has exactly 2 neighbors on left. Let $i_1 < i_2$ be the indices of these neighbors.

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Properties

- v_{i_1} is adjacent to v_{i_2}
- $i \neq j \Rightarrow \{i_1, i_2\} \neq \{j_1, j_2\}$

Outerplanar graphs

Lemma

For any vertex v_k, there are at most two vertices v_i, v_j such that $i_2 = j_2 = k$.

Collary

Edges of every outerplanar graph can be splitted into two trees $\mathbf{T_1}, \mathbf{T_2}$ such that $\Delta(\mathbf{T_2}) \leq 3$,

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Edges of every outerplanar graph can be splitted into two trees $\mathbf{T_1}, \mathbf{T_2}$ such that $\Delta(\mathbf{T_2}) \leq 3$,

Upper bound [Guan, Zhu 1999]

If $\mathbf{G}$ is an outerplanar graph, then $\chi_g(\mathbf{G}) \leq 7$.

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Edges of every outerplanar graph can be splitted into two trees $\mathbf{T_1}, \mathbf{T_2}$ such that $\Delta(\mathbf{T_2}) \leq 3$,

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If $\mathbf{G}$ is an outerplanar graph, then $\chi_g(\mathbf{G}) \leq 7$.

Lower bound [Kierstead, Trotter 1994]

There exists an outerplanar graph $\mathbf{T}$, such that $\chi_g(\mathbf{T}) = 6$

Results

Class	Lower bound	Upper bound
Forests	4	4
Cactuses	5	5
Outerplanar graphs	6	7
Planar graphs	7	17
Interval graphs	2ω	$3\omega - 2$

Open problem 1

Suppose Alice has a winning strategy for the vertex coloring game on a graph G with k colors. Does she have one for $k+1$ colors?

Open problems

Open problem 1

Suppose Alice has a winning strategy for the vertex coloring game on a graph $\mathbf{G}$ with k colors. Does she have one for $k+1$ colors?

Open problem 2

Is there a function f such that, if Alice has a winning strategy for the vertex coloring game on a graph $\mathbf{G}$ with k colors, then Alice has a winning strategy on $\mathbf{G}$ with $f(k)$.

References

Hans Bodlaender. "On the Complexity of Some Coloring Games.". In: Jan. 1990, pp. 30–40.

U. Faigle et al. "On the game chromatic number of some classes of graphs". Undefined. In: *Ars combinatoria* 35 (1993), pp. 143–150. ISSN: 0381-7032.

D. J. Guan and Xuding Zhu. "Game chromatic number of outerplanar graphs". In: *J. Graph Theory* 30 (1999), pp. 67–70.

Hal A. Kierstead and William T. Trotter. "Planar graph coloring with an uncooperative partner". In: *Journal of Graph Theory* 18 (1991), pp. 569–584.

Elżbieta Sidorowicz. "The game chromatic number and the game colouring number of cactuses". In: *Information Processing Letters* 102.4 (2007), pp. 147–151. ISSN: 0020-0190. DOI: `https://doi.org/10.1016/j.ipl.2006.12.003`.