A note on polynomials and f-factors of graphs

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Graph k-factor: a spanning subgraph that is also k-regular.

Examples:

- 1-factor of a graph is it's perfect matching.
- 2-factor of a graph is it's cycle cover.

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f-factors

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$$\forall_{v \in V(G)}\ f(v) \subseteq \{0, ..., deg(v)\}$$

Now we define f-factor as G's spanning subgraph H such that:

$$\forall_{v \in V(H)}\ deg_H(v) \in f(v)$$

Conditions for f-factor existence

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Unfortulately, if $|f(v)| > 1$ is allowed, then we do not expect to find an elegant condition that is both necessary and sufficient.

Even if $\forall_{v \in V} |f(v)| \in \{1, 2\}$, then (for some fs) deciding if there is f-factor in graph G is NP-complete by edge 3-coloring reduction (The factorization of graphs. II. L. LOVASZ).

Results from this paper

We cannot easily solve the decision problem, but we might look for conditions that are just sufficient. Authors give a following rule:

Theorem

Let $G = (V, E)$ be a graph and suppose that f satisfies:

$$|f(v)| > \lceil deg(v)/2 \rceil$$

for every $v \in V$. Then G has an f-factor.

Results from this paper p. 2

To give a second theorem we need to first define a partial f-factor, which is $\bar{f}$-factor of the same graph with $\bar{f}(v) = f(v) \cup \{0\}$.
We say that partial f-factor is non-trivial if it is not empty (there is at least one egde in the corresponding spanning subgraph).

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Now we can write down a statement of the second theorem from the paper:

Theorem

Let $G = (V, E)$ be a graph, and let f satisfy

$$|E| > \sum_{v \in V} |f(v)^c - \{0\}|$$

where $f(v)^c = \{0, 1, ..., deg(v)\} - f(v)$. Then G contains a non-trivial partial f-factor.

Combinatrial Nullstellensatz

Proofs of both theorems will take advantage of Combinatorial Nullstellensatz:

Theorem

(Combinatorial Nullstellensatz) Let $g \in \mathbb{F}[X_1, X_2, ..., X_n]$ *be a polynomial, and suppose the coefficient of the monomial* $\prod_{i=1}^{n} X_i^{t_i}$ *in* g *is non-zero, where* $t_1 + ... + t_n$ *is the total degree of* g*. Then, for any sets* $S_1, ..., S_n \subset \mathbb{F}$ *with* $|S_1| > t, |S_2|, ..., |S_n| > t_n$*, there exists* $x \in S_1 \times ... \times S_n$ *such that* $g(x) \neq 0$*.*

First theorem: proof

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First we define a polynomial:

$$g = \prod_{v \in V} \prod_{c \in f(v)^c} \left(\sum_{e \ni v} X_e - c\right)$$

Let's take $S_1 = S_2 = \ldots = S_n = \{0, 1\}$, and look at the value of X_e as a choice of including this edge into our subgraph.

First theorem: proof

$$g = \prod_{v \in V} \prod_{c \in f(v)^c} (\sum_{e \ni v} X_e - c)$$

Now we can interpret existence of $x \in S_1 \times ... \times S_n$ such that $g(x) \neq 0$ as existence of the f-factor: check if we can choose edges for our subgraph H in a way such that there is no vertex v for which $deg_H(v) \in f(v)^c$.

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So now we need to prove that in g we have a monomial of the form

$$a \prod_{e \in E} X_e^{t_e}, \; a \neq 0$$

where

$$\forall_{e \in E} t_e \in \{0, 1\}, \; \sum_{e \in E} t_e = \sum_{v \in V} |f(v)^c|$$

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First theorem: proof

To achieve this we want to show that there is a function $R : V \to 2^{\mathbb{N}}$, such that:

$$\forall_{v \in V} |R(v)| = |f(v)^c|$$

and

$$u, v \in V \wedge u \neq v \implies R(u) \cap R(v) = \emptyset$$

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Now we can choose a monomial

$$\prod_{v \in V} \prod_{e \in R(v)} X_e$$

and it will have desired properties.

First theorem: proof

How to construct such R? We can prove that there is an orientation of edges such that every vertex v has outdegree at least $\lfloor \frac{1}{2} deg(v) \rfloor$.

A simple argument is that if we add a new vertex, which is connected to vertices with odd degree, then we can orient edges as in a traverse of an Eulerian cycle.

Now, since we have $|f(v)| > \lceil deg(v)/2 \rceil$, we know that $|f(v)^c| \leq \lfloor \frac{1}{2} deg(v) \rfloor$, so R can be constructed.

Second theorem: proof

Theorem

Let $G = (V, E)$ be a graph, and let f satisfy

$$|E| > \sum_{v \in V} |f(v)^c - \{0\}|$$

where $f(v)^c = \{0, 1, ..., deg(v)\} - f(v)$. Then G contains a non-trivial partial f-factor.

Now our polynomial will be:

$$g = \prod_{v \in V} \prod_{c \in f(v)^c - \{0\}} \left(1 - \frac{\sum_{e \ni v} X_e}{c}\right) - \prod_{e \in E} (1 - X_e)$$

Second theorem: proof

$$g = \prod_{v \in V} \prod_{c \in f(v)^c - \{0\}} (1 - \frac{\sum_{e \ni v} X_e}{c}) - \prod_{e \in E} (1 - X_e)$$

Again we take $S_0 = ... = S_|E| = \{0, 1\}$. Now if there exists $x \neq 0$ such that $g(x) \neq 0$ then for this x we have that

$$\prod_{v \in V} \prod_{c \in f(v)^c - \{0\}} (1 - \frac{\sum_{e \ni v} x_e}{c}) \neq 0$$

and so

$$\sum_{v \ni e} x_e \in f(x) \cup \{0\}$$

Second theorem: proof

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Now we need to show that largest degree monomial fulfils assumptions of Combinatorial Nullstellensatz.

We know that for this graph

$$\sum_{v \in V} |f(v)^c - \{0\}| < |E|$$

So it's obvious that the largest monomial is just

$$(-1)^{|E|+1} \prod_{e \in E} X_e$$