Multiple list colouring of planar graphs

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Definition

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Fractional chromatic number of G

$$\chi_f(G) = inf\{\frac{a}{b}: G \text{ is (a,b)-colourable}\}.$$

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a-list assignment of G *is a mapping* L *which assigns to each vertex* v *a set* $L(v)$ *of* a *permissible colours.*

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Fractional choice number of G

$$ch_f(G) = inf\{\frac{a}{b} : G \text{ is (a,b)-choosable}\}.$$

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Theorem

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For each positive integer m*, there is a planar graph* G *which is not* $(4m + \lfloor \frac{2m-1}{9} \rfloor, m)$*-choosable.*

Proof

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For each positive integer m, there is a planar graph G which is not $(4m + \lfloor \frac{2m-1}{9} \rfloor, m)$-choosable.

Let m be the fixed positive integer and $k = \lfloor \frac{2m-1}{9} \rfloor$. To proof the theorem we will show the construction of a planar graph H which is not $(4m + k, m)$-choosable.

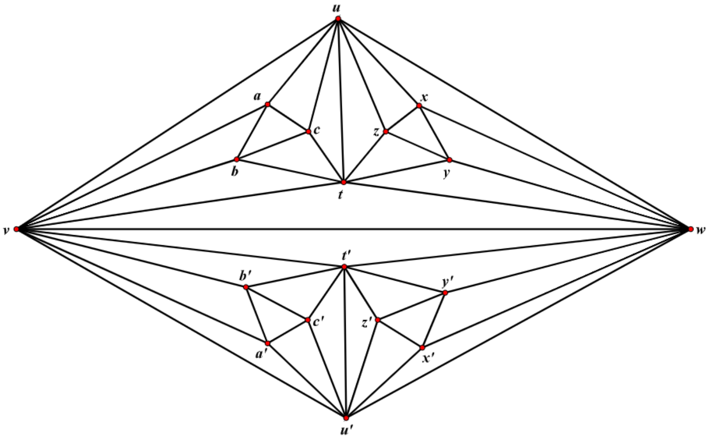

Fig. 1. The graph G.

Lemma

Lemma

Let G be a graph shown above. Let A and B be disjoint sets, such that $|A| = |B| = m$. Let L be a list assignment of G for which the following hold:

1. $|L(s)| = 4m + k$ *for each vertex* s*, except that* $L(u) = A, L(u') = B$.
2. *There is no* m*-fold* L*-colouring of* G.

Proof of Lemma

Let A, B be any disjoint sets of colours such that $|A| = |B| = m$. Let C, D be any disjoint sets of colours such that $|C| = |D| = 2m + k$ and C, D are disjoint from both A and B.

Let $X, X' \subseteq C$ be disjoint subsets such that $|X| = |X'| = m$. L will be defined in the following way:

- $L(u) = A$ and $L(u') = B$.
- $L(v) = L(w) = L(t) = L(t') = A \cup B \cup C$.
- $L(x) = L(a) = X \cup A \cup D$ and $L(x') = L(a') = X' \cup A \cup D$.
- $L(y) = L(b) = X \cup B \cup D$ and $L(y') = L(b') = X' \cup A \cup D$.
- $L(z) = L(c) = L(z') = L(c') = A \cup B \cup D$.

Now we will show the second property of the L - there is no m-fold L-colouring of G.
Lets assume that ϕ is an m-fold L-colouring of G.
Then $\phi(u) = A$ and $\phi(u') = B$ and $\phi(v), \phi(w)$ are disjoint m-subsets of C. So

$$|(\phi(v) \cup \phi(w)) \cap (X \cup X')| \geq 2m - k.$$

By symmetry of (u, v, w) and (u', v, w), we can assume that

$$|(\phi(v) \cup \phi(w)) \cap X| \geq |(\phi(v) \cup \phi(w)) \cap X'|.$$

So

$$|\phi(v) \cap X| + |\phi(w) \cap X| = |(\phi(v) \cup \phi(w)) \cap X| \geq m - \frac{k}{2}.$$

By symmetry of (u, v, t) and (u, w, t), we can assume that

$$|\phi(v) \cap X| \geq |\phi(w) \cap X|,$$

so

$$|\phi(v) \cap X| \geq \frac{m}{2} - \frac{k}{4}.$$

Let $T = X - \phi(v)$. We have

$$|T| = |X| - |X \cap \phi(v)| \leq \frac{m}{2} + \frac{k}{4}.$$

Let $R = B - \phi(t)$ and $S = C - (\phi(v) \cup \phi(w))$. Then $|S| \leq k$. As $\phi(t)$ is disjoint from $\phi(u) \cup \phi(v) \cup \phi(w)$, we know that $\phi(t) \subseteq B \cup S$. Hence

$$|R| \leq |S| = k.$$

By deleting the colours used by the neighbours of a, b, c, respectively, we have

- $\phi(a) \subseteq D \cup T$,
- $\phi(b) \subseteq D \cup R \cup T$,
- $\phi(c) \subseteq D \cup R$.

As $\phi(a), \phi(b), \phi(c)$ are pairwise disjoint, we have

$$3m = |\phi(a) \cup \phi(b) \cup \phi(c)| \leq |D| + |T| + |R|$$

$$\leq (2m + k) + (\frac{m}{2} + \frac{k}{4}) + k = \frac{5m}{2} + \frac{9k}{4} < 3m,$$

a contradiction.

Back to proof of Theorem

Let $p = \binom{4m+k}{m,m,2m+k}$, and let G be obtained from the disjoint union of p copies of H by identifying all the copies of u into a single vertex (also named as u) and all the copies of u' into a single vertex (also named as u'), and then add an edge connecting u and u'. For sure G is a planar graph.

To show that G is not $(4m+k, m)$-choosable, let Z be a set of $4m+k$ colours. Let $L(u) = L(u') = Z$. There are p possible m-fold L-colourings of u and u'. Each such colouring ϕ corresponds to one copy of H. In that copy of H, define the list assignment as in the proof of Lemma, by replacing A with $\phi(u)$ and B with $\phi(u')$. Now Lemma implies that no m-fold colouring of u and u' can be extended to and m-fold L-colouring of G. $\square$

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Thomassen proved that every planar graph is 5-choosable. It is possible to adopt proof and show for any positive integer m, every planar graph $(5m, m)$-choosable. Given a positive integer m, let $a(m)$ be the minimum integer such that every planar graph is $(a(m), m)$-choosable. Combining Thomassen's result and Theorem of this paper, we have

$$4m + \lfloor \frac{2m-1}{9} \rfloor + 1 \leq a(m) \leq 5m.$$

For $m = 1$, the upper bound and the lower bound coincide. So $a(1) = 5$. As m becomes bigger, the gap between the upper and lower bounds increases. A natural question is what is the exact value of $a(m)$. Authors conjecture that the upper bound is not always tight.

Conjecture

There is a constant integer m *such that every planar graph is* $(5m-1, m)$*-choosable.*

Conjecture

Every planar graph is $(9, 2)$*-choosable.*

Conjecture

(By Erdos, Rubin and Taylor) If G *is (a,b)-choosable, then for any positive integer* m, G *is* (am, bm)*-choosable*

Definition

G is strongly α***-choosable*** *if for any positive integer* m*,* G *is* $(\lceil \alpha m \rceil, m)$*-choosable.*

Definition

Strong choice number of G *is*

$$ch_s(G) = inf\{\alpha\text{: G is strongly } \alpha\text{-choosable.}\}$$

- Is the infimum in the definition of $ch_s(G)$ always attained (and hence can be replaced by the minimum)?
- What real numbers are the strong choice number of graphs?
- Is $ch_s(G)$ rational for all finite graphs?