Can a party represent its constituency?

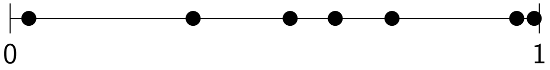
Marcin Serwin

Uniwersytet Jagielloński

2022-04-07

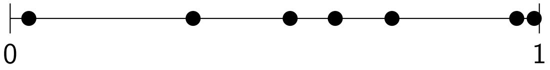
Political model

We will represent political interests on a one dimensional axis from 0 to 1. Each party member will be represented as a number in this interval.



Political model

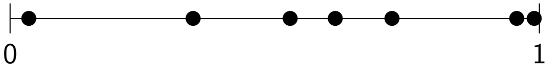
We will represent political interests on a one dimensional axis from 0 to 1. Each party member will be represented as a number in this interval.



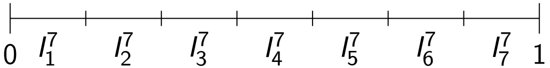
We assume that party's interests are distributed more or less uniformly, thus our candidates should also be so.

Political model

We will represent political interests on a one dimensional axis from 0 to 1. Each party member will be represented as a number in this interval.

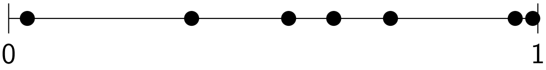


We assume that party's interests are distributed more or less uniformly, thus our candidates should also be so. Formally, for a given k, we divide the axis into k equal buckets $I_1^k, \ldots, I_k^k$:

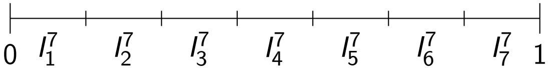


Political model

We will represent political interests on a one dimensional axis from 0 to 1. Each party member will be represented as a number in this interval.



We assume that party's interests are distributed more or less uniformly, thus our candidates should also be so. Formally, for a given k, we divide the axis into k equal buckets $I_1^k, \ldots, I_k^k$:



We say that $x_1, x_2, \ldots, x_k$ is a *representative body* if for each I_i^k there exist x_j such that $x_j \in I_i^k$.

Political model continued

Our parties will partake in elections. If there are n seats in the parliament, and the party receives p% votes then they will receive np% seats.

Political model continued

Our parties will partake in elections. If there are n seats in the parliament, and the party receives $p\%$ votes then they will receive $np\%$ seats. In our model parties submit ordered list of candidates $(x_1, \ldots, x_n)$ before the election and if the party wins k seats, then candidates $x_1, \ldots, x_k$ will become the representatives.

Political model continued

Our parties will partake in elections. If there are n seats in the parliament, and the party receives $p\%$ votes then they will receive $np\%$ seats. In our model parties submit ordered list of candidates $(x_1, \dots, x_n)$ before the election and if the party wins k seats, then candidates $x_1, \dots, x_k$ will become the representatives.
This motivates us to define a *representative list*, that is, an ordered tuple $(x_1, \dots, x_n)$, such that for each $k \in \{1, \dots, n\}$, $x_1, \dots, x_k$ is a representative body [Kat84].

Political model continued

Our parties will partake in elections. If there are n seats in the parliament, and the party receives $p\%$ votes then they will receive $np\%$ seats. In our model parties submit ordered list of candidates $(x_1, \ldots, x_n)$ before the election and if the party wins k seats, then candidates $x_1, \ldots, x_k$ will become the representatives.

This motivates us to define a *representative list*, that is, an ordered tuple $(x_1, \ldots, x_n)$, such that for each $k \in \{1, \ldots, n\}$, $x_1, \ldots, x_k$ is a representative body [Kat84].

This ensures that regardless of how many candidates will be appointed, the interests will be more or less represented.

Political model continued

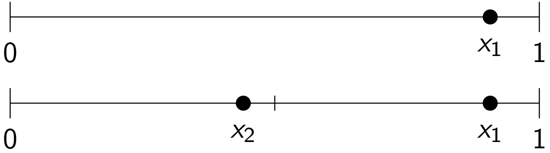
Our parties will partake in elections. If there are n seats in the parliament, and the party receives $p\%$ votes then they will receive $np\%$ seats. In our model parties submit ordered list of candidates $(x_1, \ldots, x_n)$ before the election and if the party wins k seats, then candidates $x_1, \ldots, x_k$ will become the representatives.
This motivates us to define a *representative list*, that is, an ordered tuple $(x_1, \ldots, x_n)$, such that for each $k \in \{1, \ldots, n\}$, $x_1, \ldots, x_k$ is a representative body [Kat84].
This ensures that regardless of how many candidates will be appointed, the interests will be more or less represented.

Political model continued

Our parties will partake in elections. If there are n seats in the parliament, and the party receives $p\%$ votes then they will receive $np\%$ seats. In our model parties submit ordered list of candidates $(x_1, \ldots, x_n)$ before the election and if the party wins k seats, then candidates $x_1, \ldots, x_k$ will become the representatives.

This motivates us to define a *representative list*, that is, an ordered tuple $(x_1, \ldots, x_n)$, such that for each $k \in \{1, \ldots, n\}$, $x_1, \ldots, x_k$ is a representative body [Kat84].

This ensures that regardless of how many candidates will be appointed, the interests will be more or less represented.

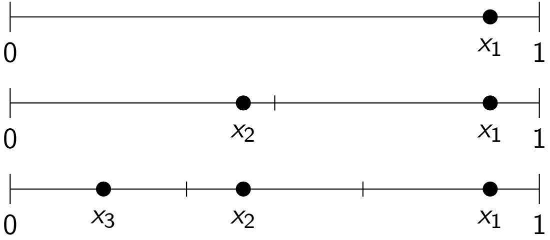


Political model continued

Our parties will partake in elections. If there are n seats in the parliament, and the party receives $p\%$ votes then they will receive $np\%$ seats. In our model parties submit ordered list of candidates $(x_1, \dots, x_n)$ before the election and if the party wins k seats, then candidates $x_1, \dots, x_k$ will become the representatives.

This motivates us to define a *representative list*, that is, an ordered tuple $(x_1, \dots, x_n)$, such that for each $k \in \{1, \dots, n\}$, $x_1, \dots, x_k$ is a representative body [Kat84].

This ensures that regardless of how many candidates will be appointed, the interests will be more or less represented.



Problem

Can we always construct a representative list?

Problem

Can we always construct a representative list? It turns out that that the answer is **NO**.

Problem

Can we always construct a representative list? It turns out that that the answer is **NO**.

Theorem

A party can construct representative list if and only if $n \leqslant 17$.

Problem

Can we always construct a representative list? It turns out that that the answer is **NO**.

Theorem

A party can construct representative list if and only if $n \leqslant 17$.

Beginning of proof by [War76]:

Proof sketch

Let's assume that there exists a representative list $(x_1, \ldots, x_{18})$. Let x_k^n denote a number lying in the interval $\left[\frac{k-1}{n}, \frac{k}{n}\right)$ in the sequence $(x_1, \ldots, x_n)$.

Problem

Can we always construct a representative list? It turns out that that the answer is **NO**.

Theorem

A party can construct representative list if and only if $n \leqslant 17$.

Beginning of proof by [War76]:

Proof sketch

Let's assume that there exists a representative list $(x_1, \ldots, x_{18})$. Let x_k^n denote a number lying in the interval $\left[\frac{k-1}{n}, \frac{k}{n}\right)$ in the sequence $(x_1, \ldots, x_n)$. Let us consider all possible values of x_5^9. By symmetry we only need to consider:

- $\frac{4}{9} \leqslant x_5^9 < \frac{5}{11}$
- $\frac{5}{11} \leqslant x_5^9 < \frac{6}{13}$
- $\frac{6}{13} \leqslant x_5^9 < \frac{7}{15}$
- $\frac{7}{15} \leqslant x_5^9 < \frac{8}{17}$
- $\frac{8}{17} \leqslant x_5^9 < \frac{1}{2}$
- $x_5^9 = \frac{1}{2}$

Proof continued

Proof sketch (continued)

Let's consider the first case that is $\frac{4}{9} \leqslant x_5^9 < \frac{5}{11}$.

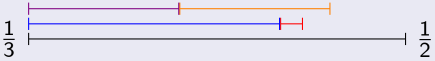
Proof continued

Proof sketch (continued)

Let's consider the first case that is $\frac{4}{9} \leqslant x_5^9 < \frac{5}{11}$.

We can deduce that:

1. $$x_5^9 = x_5^{11} = x_7^{14} = x_7^{15} = x_8^{16} = x_8^{17} \implies x_4^9 = x_6^{15}$$

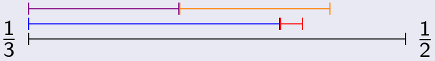


Proof continued

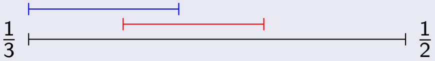
Proof sketch (continued)

Let's consider the first case that is $\frac{4}{9} \leqslant x_5^9 < \frac{5}{11}$.
We can deduce that:

1. $$x_5^9 = x_5^{11} = x_7^{14} = x_7^{15} = x_8^{16} = x_8^{17} \implies x_4^9 = x_6^{15}$$



2. $$\frac{3}{8} \leqslant x_6^{14} = x_7^{17} = x_7^{16} = x_6^{15} < \frac{2}{5} \implies \frac{3}{8} \leqslant x_4^9 < \frac{2}{5}$$

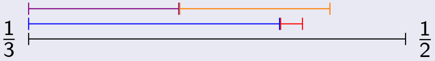


Proof continued

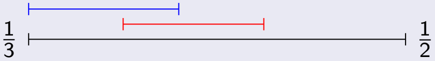
Proof sketch (continued)

Let's consider the first case that is $\frac{4}{9} \leqslant x_5^9 < \frac{5}{11}$.
We can deduce that:

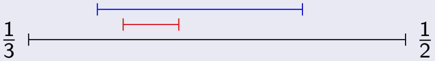
1. $$x_5^9 = x_5^{11} = x_7^{14} = x_7^{15} = x_8^{16} = x_8^{17} \implies x_4^9 = x_6^{15}$$



2. $$\frac{3}{8} \leqslant x_6^{14} = x_7^{17} = x_7^{16} = x_6^{15} < \frac{2}{5} \implies \frac{3}{8} \leqslant x_4^9 < \frac{2}{5}$$



3. $$x_4^9 = x_5^{11} \implies x_4^9 = x_5^9$$



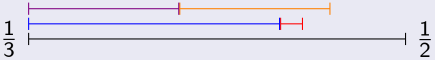
which is contradictory.

Proof continued

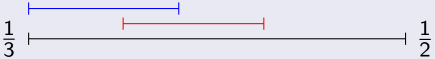
Proof sketch (continued)

Let's consider the first case that is $\frac{4}{9} \leqslant x_5^9 < \frac{5}{11}$.
We can deduce that:

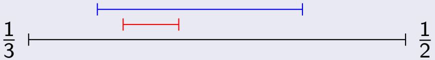
1. $$x_5^9 = x_5^{11} = x_7^{14} = x_7^{15} = x_8^{16} = x_8^{17} \implies x_4^9 = x_6^{15}$$



2. $$\frac{3}{8} \leqslant x_6^{14} = x_7^{17} = x_7^{16} = x_6^{15} < \frac{2}{5} \implies \frac{3}{8} \leqslant x_4^9 < \frac{2}{5}$$



3. $$x_4^9 = x_5^{11} \implies x_4^9 = x_5^9$$



which is contradictory. Other cases can be solved analogously. □

Proof continued

What remains is to show that it is possible to do so with 17 candidates. We can construct such list by placing the following constraints:

$$\frac{4}{7} \leqslant x_1 < \frac{7}{12} \qquad \frac{2}{7} \leqslant x_2 < \frac{5}{17} \qquad \frac{16}{17} \leqslant x_3 < 1 \qquad \frac{1}{14} \leqslant x_4 < \frac{1}{13} \qquad \frac{8}{11} \leqslant x_5 < \frac{11}{15}$$

$$\frac{5}{11} \leqslant x_6 < \frac{6}{13} \qquad \frac{1}{7} \leqslant x_7 < \frac{2}{13} \qquad \frac{14}{17} \leqslant x_8 < \frac{5}{6} \qquad \frac{3}{8} \leqslant x_9 < \frac{5}{13} \qquad \frac{11}{17} \leqslant x_{10} < \frac{2}{3}$$

$$\frac{3}{14} \leqslant x_{11} < \frac{3}{13} \qquad \frac{15}{17} \leqslant x_{12} < \frac{11}{12} \qquad \frac{1}{2} \leqslant x_{13} < \frac{9}{17} \qquad 0 \leqslant x_{14} < \frac{1}{17} \qquad \frac{13}{17} \leqslant x_{15} < \frac{4}{5}$$

$$\frac{5}{16} \leqslant x_{16} < \frac{6}{17} \qquad \frac{10}{17} \leqslant x_{17} < \frac{11}{17}$$

Proof continued

What remains is to show that it is possible to do so with 17 candidates. We can construct such list by placing the following constraints:

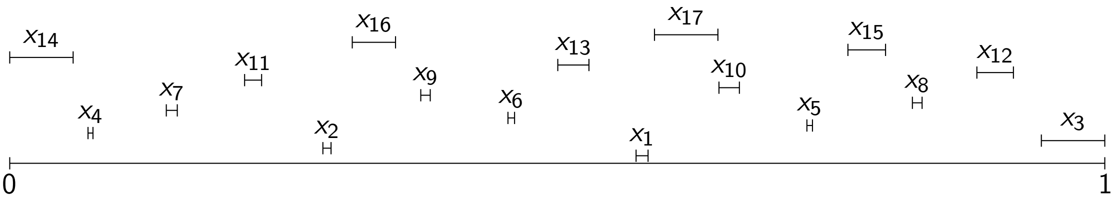
$$\frac{4}{7} \leqslant x_1 < \frac{7}{12} \quad \frac{2}{7} \leqslant x_2 < \frac{5}{17} \quad \frac{16}{17} \leqslant x_3 < 1 \quad \frac{1}{14} \leqslant x_4 < \frac{1}{13} \quad \frac{8}{11} \leqslant x_5 < \frac{11}{15}$$

$$\frac{5}{11} \leqslant x_6 < \frac{6}{13} \quad \frac{1}{7} \leqslant x_7 < \frac{2}{13} \quad \frac{14}{17} \leqslant x_8 < \frac{5}{6} \quad \frac{3}{8} \leqslant x_9 < \frac{5}{13} \quad \frac{11}{17} \leqslant x_{10} < \frac{2}{3}$$

$$\frac{3}{14} \leqslant x_{11} < \frac{3}{13} \quad \frac{15}{17} \leqslant x_{12} < \frac{11}{12} \quad \frac{1}{2} \leqslant x_{13} < \frac{9}{17} \quad 0 \leqslant x_{14} < \frac{1}{17} \quad \frac{13}{17} \leqslant x_{15} < \frac{4}{5}$$

$$\frac{5}{16} \leqslant x_{16} < \frac{6}{17} \quad \frac{10}{17} \leqslant x_{17} < \frac{11}{17}$$

Visually:



Additional conclusions

Indeed the presented solution for $n = 17$ is just one among 768 (1536 if we count symmetrical solutions separately) [War76].

Additional conclusions

Indeed the presented solution for $n = 17$ is just one among 768 (1536 if we count symmetrical solutions separately) [War76]. Even though there are many solutions, all of them must satisfy some rather strong restrictions such as:

- $\frac{3}{8} \leqslant x_9 < \frac{5}{13}$
- $\frac{9}{14} \leqslant x_{10} < \frac{2}{3}$
- $\frac{3}{14} \leqslant x_{11} < \frac{3}{13}$
- $\frac{1}{2} \leqslant x_{13} < \frac{9}{17}$
- $\frac{5}{16} \leqslant x_{16} < \frac{6}{17}$

Additional conclusions

Indeed the presented solution for $n = 17$ is just one among 768 (1536 if we count symmetrical solutions separately) [War76]. Even though there are many solutions, all of them must satisfy some rather strong restrictions such as:

- $\frac{3}{8} \leqslant x_9 < \frac{5}{13}$
- $\frac{9}{14} \leqslant x_{10} < \frac{2}{3}$
- $\frac{3}{14} \leqslant x_{11} < \frac{3}{13}$
- $\frac{1}{2} \leqslant x_{13} < \frac{9}{17}$
- $\frac{5}{16} \leqslant x_{16} < \frac{6}{17}$

This explains why the number of solutions suddenly drops to zero when going from 17 to 18.

Looser constraints

Throughout this presentation, we assumed that each candidate must feel exactly one bucket.

Looser constraints

Throughout this presentation, we assumed that each candidate must feel exactly one bucket. We may wonder what happens if we loosen this restriction and allow some additional candidates to fill the buckets.

Looser constraints

Throughout this presentation, we assumed that each candidate must feel exactly one bucket. We may wonder what happens if we loosen this restriction and allow some additional candidates to fill the buckets. Formally for a given d we want to consider set $x_1, \ldots, x_{n+d}$ as d-representative body if for each I_i^n there exists x_j in the set such that $x_j \in I_i^n$.

Looser constraints

Throughout this presentation, we assumed that each candidate must feel exactly one bucket. We may wonder what happens if we loosen this restriction and allow some additional candidates to fill the buckets. Formally for a given d we want to consider set $x_1, \ldots, x_{n+d}$ as d-representative body if for each I_i^n there exists x_j in the set such that $x_j \in I_i^n$. Analogously we consider list d-representative if all its prefixes are d-representative bodies.

Looser constraints

Throughout this presentation, we assumed that each candidate must feel exactly one bucket. We may wonder what happens if we loosen this restriction and allow some additional candidates to fill the buckets. Formally for a given d we want to consider set $x_1, \ldots, x_{n+d}$ as d-representative body if for each I_i^n there exists x_j in the set such that $x_j \in I_i^n$. Analogously we consider list d-representative if all its prefixes are d-representative bodies.
It turns out that while the longest representative list increases significantly, it is still finite.

Looser constraints

Throughout this presentation, we assumed that each candidate must feel exactly one bucket. We may wonder what happens if we loosen this restriction and allow some additional candidates to fill the buckets. Formally for a given d we want to consider set $x_1, \ldots, x_{n+d}$ as d-representative body if for each I_i^n there exists x_j in the set such that $x_j \in I_i^n$. Analogously we consider list d-representative if all its prefixes are d-representative bodies.
It turns out that while the longest representative list increases significantly, it is still finite. As proven by [BG70]

Theorem

The length of longest d-representative list is strictly smaller than $4^{(d+2)^2}$.

Better bounds

Let us call the length of longest d-representative list $s(d)$.

Better bounds

Let us call the length of longest d-representative list $s(d)$. To restate previous theorems: $s(0) = 17$ and $s(d) < 4^{(d+2)^2}$.

Better bounds

Let us call the length of longest d-representative list $s(d)$. To restate previous theorems: $s(0) = 17$ and $s(d) < 4^{(d+2)^2}$. It is also known that $s(1) \geqslant 31$ [Lev20]. It's been claimed that $s(1) = 31$ [Sil], but it is not as of yet verified.

Better bounds

Let us call the length of longest d-representative list $s(d)$. To restate previous theorems: $s(0) = 17$ and $s(d) < 4^{(d+2)^2}$. It is also known that $s(1) \geqslant 31$ [Lev20]. It's been claimed that $s(1) = 31$ [Sil], but it is not as of yet verified. While the [BG70] result was first to establish the upper bound, it has been succesfully improved in the following years.

Better bounds

Let us call the length of longest d-representative list $s(d)$. To restate previous theorems: $s(0) = 17$ and $s(d) < 4^{(d+2)^2}$. It is also known that $s(1) \geqslant 31$ [Lev20]. It's been claimed that $s(1) = 31$ [Sil], but it is not as of yet verified. While the [BG70] result was first to establish the upper bound, it has been succesfully improved in the following years.

- In 2013 Graham showed that $s(d) = O(d^3)$ [Gra13].

Better bounds

Let us call the length of longest d-representative list $s(d)$. To restate previous theorems: $s(0) = 17$ and $s(d) < 4^{(d+2)^2}$. It is also known that $s(1) \geqslant 31$ [Lev20]. It's been claimed that $s(1) = 31$ [Sil], but it is not as of yet verified. While the [BG70] result was first to establish the upper bound, it has been succesfully improved in the following years.

- In 2013 Graham showed that $s(d) = O(d^3)$ [Gra13].
- In 2020 Levy showed that $s(d) < 24801d^3 + 942d^2 + 3$ for $d \geqslant 1$ [Lev20].

Better bounds

Let us call the length of longest d-representative list $s(d)$. To restate previous theorems: $s(0) = 17$ and $s(d) < 4^{(d+2)^2}$. It is also known that $s(1) \geqslant 31$ [Lev20]. It's been claimed that $s(1) = 31$ [Sil], but it is not as of yet verified. While the [BG70] result was first to establish the upper bound, it has been succesfully improved in the following years.

- In 2013 Graham showed that $s(d) = O(d^3)$ [Gra13].
- In 2020 Levy showed that $s(d) < 24801d^3 + 942d^2 + 3$ for $d \geqslant 1$ [Lev20]. This paper also showed the first known lower bound namely that $s(d) \geqslant \lfloor \sqrt{4d + 895} \rfloor + 1$ for $d \geqslant 1$.

Better bounds

Let us call the length of longest d-representative list $s(d)$. To restate previous theorems: $s(0) = 17$ and $s(d) < 4^{(d+2)^2}$. It is also known that $s(1) \geqslant 31$ [Lev20]. It's been claimed that $s(1) = 31$ [Sil], but it is not as of yet verified. While the [BG70] result was first to establish the upper bound, it has been succesfully improved in the following years.

- In 2013 Graham showed that $s(d) = O(d^3)$ [Gra13].
- In 2020 Levy showed that $s(d) < 24801d^3 + 942d^2 + 3$ for $d \geqslant 1$ [Lev20]. This paper also showed the first known lower bound namely that $s(d) \geqslant \lfloor \sqrt{4d + 895} \rfloor + 1$ for $d \geqslant 1$.
- The newest result obtained by Konyagin in 2021 improves it further by showing that $s(d) \geqslant 2d$ for $d \geqslant 1$, and $s(d) < 200d$ for sufficiently large d [Kon21].

A different point of view

So far we have dealt exclusively with the assumption that at least one point must be in each interval and the intervals are predefined.

A different point of view

So far we have dealt exclusively with the assumption that at least one point must be in each interval and the intervals are predefined. We may wonder what happens if we loosen the former restriction and make the latter more restrictive.

A different point of view

So far we have dealt exclusively with the assumption that at least one point must be in each interval and the intervals are predefined. We may wonder what happens if we loosen the former restriction and make the latter more restrictive. To formalize:

Problem

For a given K, is there an infinite sequence $(x_1, x_2, \ldots) \in [0,1)^{\mathbb{N}}$ such that for any $n \in \mathbb{N}$ and for any two subintervals $I, J \subseteq [0,1)$ it holds that

$$||\{x_i | x_i \in I \wedge i < n\}| - |\{x_i | x_i \in J \wedge i < n\}|| \leqslant K$$

A different point of view

So far we have dealt exclusively with the assumption that at least one point must be in each interval and the intervals are predefined. We may wonder what happens if we loosen the former restriction and make the latter more restrictive. To formalize:

Problem

For a given K, is there an infinite sequence $(x_1, x_2, \ldots) \in [0,1)^{\mathbb{N}}$ such that for any $n \in \mathbb{N}$ and for any two subintervals $I, J \subseteq [0,1)$ it holds that

$$||\{x_i | x_i \in I \wedge i < n\}| - |\{x_i | x_i \in J \wedge i < n\}|| \leqslant K$$

It turns out that the answer is once again **NO** as was proven by van Aardenne-Ehrenfest in 1945 [Aar45].

Improvement

The mentioned result was subsequently improved by her in 1949 [Aar49] to state that

Theorem

For large enough N and a sequence $(x_1, \ldots, x_n) \in [0,1)^N$, there exist $n \leqslant N$ and $\alpha \in [0,1)$ such that

$$||\{x_i | x_i < \alpha \wedge i < n\}| - n\alpha| > c \cdot \frac{\log \log N}{\log \log \log N}$$

where c is a positive constant.

Improvement

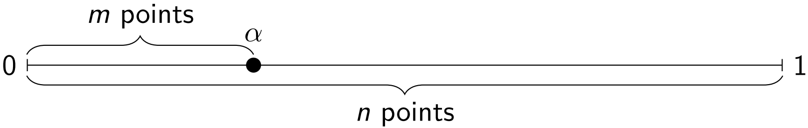
The mentioned result was subsequently improved by her in 1949 [Aar49] to state that

Theorem

For large enough N and a sequence $(x_1, \ldots, x_n) \in [0,1)^N$, there exist $n \leqslant N$ and $\alpha \in [0,1)$ such that

$$||\{x_i | x_i < \alpha \wedge i < n\}| - n\alpha| > c \cdot \frac{\log \log N}{\log \log \log N}$$

where c is a positive constant.



$$|m - n\alpha| > c \cdot \frac{\log \log N}{\log \log \log N}$$

Further improvements

This result was later improved twice:

Further improvements

This result was later improved twice:

- First by Roth in 1954 [Rot54]: $||\{x_i|x_i < \alpha \wedge i < n\}| - n\alpha| > c\sqrt{\log n}$

Further improvements

This result was later improved twice:

- First by Roth in 1954 [Rot54]: $||\{x_i|x_i < \alpha \wedge i < n\}| - n\alpha| > c\sqrt{\log n}$
- Then by Schmidt in 1972 [Sch72]: $||\{x_i|x_i < \alpha \wedge i < n\}| - n\alpha| > c \log n$

Further improvements

This result was later improved twice:

- First by Roth in 1954 [Rot54]: $||\{x_i|x_i < \alpha \wedge i < n\}| - n\alpha| > c\sqrt{\log n}$
- Then by Schmidt in 1972 [Sch72]: $||\{x_i|x_i < \alpha \wedge i < n\}| - n\alpha| > c \log n$

It turns out that $o(\log n)$ is the best possible result, since back in 1904 Lerch [Ler04] showed that for any irrational α such that its continued fraction has bounded denominators, the sequence

$$\{\alpha\}, \{2\alpha\}, \{3\alpha\}, \ldots$$

where $\{x\} = x - \lfloor x \rfloor$,

Further improvements

This result was later improved twice:

- First by Roth in 1954 [Rot54]: $||\{x_i|x_i < \alpha \wedge i < n\}| - n\alpha| > c\sqrt{\log n}$
- Then by Schmidt in 1972 [Sch72]: $||\{x_i|x_i < \alpha \wedge i < n\}| - n\alpha| > c \log n$

It turns out that $o(\log n)$ is the best possible result, since back in 1904 Lerch [Ler04] showed that for any irrational α such that its continued fraction has bounded denominators, the sequence

$$\{\alpha\}, \{2\alpha\}, \{3\alpha\}, \ldots$$

where $\{x\} = x - \lfloor x \rfloor$, satisfies

$$||\{x_i|x_i < \alpha \wedge i < n\}| - n\alpha| < c \log n$$

Higher dimensions

One of the nice features of the alternative version we introduced is that it can be easily generalized into dimensions higher than one.

Higher dimensions

One of the nice features of the alternative version we introduced is that it can be easily generalized into dimensions higher than one. Let's consider the case of 2:

Definition

Let $P_1, \ldots, P_N \in [0,1)^2$ be points. Now define $S(x,y)$ for $x, y \in [0,1)$ be the number of points u, v such that $u < x \wedge v < v$.

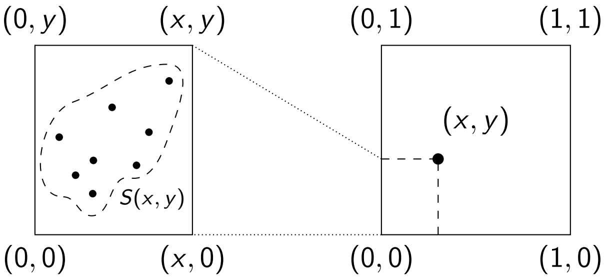
Higher dimensions

One of the nice features of the alternative version we introduced is that it can be easily generalized into dimensions higher than one. Let's consider the case of 2:

Definition

Let $P_1, \ldots, P_N \in [0,1)^2$ be points. Now define $S(x,y)$ for $x, y \in [0,1)$ be the number of points u, v such that $u < x \wedge v < v$.

Visually:



2D version

Can we say anything about the value of $S(x, y)$?

2D version

Can we say anything about the value of $S(x, y)$? It turns out that any one dimensional bound also implies a similar two dimensional bound [Rot54].

2D version

Can we say anything about the value of $S(x, y)$? It turns out that any one dimensional bound also implies a similar two dimensional bound [Rot54]. Thus by again using theorem in [Sch72], we get

Theorem

Given (not necessarily distinct) points $P_1, \ldots, P_N \in [0, 1)^2$, there exists x, y such that

$$|S(x, y) - Nxy| > c \cdot \log N$$

for a positive constant c.

2D version

Can we say anything about the value of $S(x, y)$? It turns out that any one dimensional bound also implies a similar two dimensional bound [Rot54]. Thus by again using theorem in [Sch72], we get

Theorem

Given (not necessarily distinct) points $P_1, \ldots, P_N \in [0, 1)^2$, there exists x, y such that

$$|S(x, y) - Nxy| > c \cdot \log N$$

for a positive constant c.

Let us define $d(\mathbf{P}, n) = |S_{\mathbf{P}}(x, y) - nxy|$ for $\mathbf{P} = (P_1, P_2, \ldots)$.

2D version

Can we say anything about the value of $S(x, y)$? It turns out that any one dimensional bound also implies a similar two dimensional bound [Rot54]. Thus by again using theorem in [Sch72], we get

Theorem

Given (not necessarily distinct) points $P_1, \ldots, P_N \in [0, 1)^2$, there exists x, y such that

$$|S(x, y) - Nxy| > c \cdot \log N$$

for a positive constant c.

Let us define $d(\mathbf{P}, n) = |S_{\mathbf{P}}(x, y) - nxy|$ for $\mathbf{P} = (P_1, P_2, \ldots)$. From the previous theorem we know that there exists $\mathbf{P}$ such that

$$\limsup_{n \to \infty} d(\mathbf{P}, n) = \infty$$

2D version continued

Given this we may wonder whether there exists **P** such that

$$\limsup_{n\to\infty} \frac{d(\mathbf{P}, n)}{\log n} < \infty$$

2D version continued

Given this we may wonder whether there exists **P** such that

$$\limsup_{n\to\infty} \frac{d(\mathbf{P}, n)}{\log n} < \infty$$

It turns out that it is indeed the case.

2D version continued

Given this we may wonder whether there exists **P** such that

$$\limsup_{n\to\infty} \frac{d(\mathbf{P}, n)}{\log n} < \infty$$

It turns out that it is indeed the case. In [Bec89] it was shown that in two-dimensional case

$$d(\mathbf{P}, n) > (\log n)(\log\log n)^{\frac{1}{8}-\varepsilon}$$

for infinitely many n.

2D version continued

Given this we may wonder whether there exists **P** such that

$$\limsup_{n\to\infty} \frac{d(\mathbf{P}, n)}{\log n} < \infty$$

It turns out that it is indeed the case. In [Bec89] it was shown that in two-dimensional case

$$d(\mathbf{P}, n) > (\log n)(\log \log n)^{\frac{1}{8}-\varepsilon}$$

for infinitely many n. The general conjecture for higher dimensions i.e. whether in dimension k there are **P** such that

$$\limsup_{n\to\infty} \frac{d(\mathbf{P}, n)}{(\log n)^{k-1}} < \infty$$

is still open.

Conjectures

While the previous theorem established that for $k = 2$

$$d(\mathbf{P}, n) > (\log n)(\log \log n)^{\frac{1}{8}-\varepsilon}$$

which is probably not the best possible lower bound.

Conjectures

While the previous theorem established that for $k = 2$

$$d(\mathbf{P}, n) > (\log n)(\log \log n)^{\frac{1}{8}-\varepsilon}$$

which is probably not the best possible lower bound. It is currently conjectured that in k dimensions it is the case that

$$d(\mathbf{P}, n) > c_k(\log n)^k$$

Conjectures

While the previous theorem established that for $k = 2$

$$d(\mathbf{P}, n) > (\log n)(\log\log n)^{\frac{1}{8}-\varepsilon}$$

which is probably not the best possible lower bound. It is currently conjectured that in k dimensions it is the case that

$$d(\mathbf{P}, n) > c_k(\log n)^k$$

In [Rot54] it was proven that $d(\mathbf{P}, n) > c_k(\log n)^{\frac{k}{2}}$.

Conjectures

While the previous theorem established that for $k = 2$

$$d(\mathbf{P}, n) > (\log n)(\log \log n)^{\frac{1}{8}-\varepsilon}$$

which is probably not the best possible lower bound. It is currently conjectured that in k dimensions it is the case that

$$d(\mathbf{P}, n) > c_k(\log n)^k$$

In [Rot54] it was proven that $d(\mathbf{P}, n) > c_k(\log n)^{\frac{k}{2}}$. This result was improved in [BLV07] to state that there exists function $\eta(k) > 0$ such that $d(\mathbf{P}, n) > c_k(\log n)^{\frac{k}{2}+\eta(k)}$.

Conjectures

While the previous theorem established that for $k = 2$

$$d(\mathbf{P}, n) > (\log n)(\log \log n)^{\frac{1}{8}-\varepsilon}$$

which is probably not the best possible lower bound. It is currently conjectured that in k dimensions it is the case that

$$d(\mathbf{P}, n) > c_k(\log n)^k$$

In [Rot54] it was proven that $d(\mathbf{P}, n) > c_k(\log n)^{\frac{k}{2}}$. This result was improved in [BLV07] to state that there exists function $\eta(k) > 0$ such that $d(\mathbf{P}, n) > c_k(\log n)^{\frac{k}{2}+\eta(k)}$. This slow progress prompted some to speculate that the actual conjecture should be weaker, i.e.

$$d(\mathbf{P}, n) > c_k(\log n)^{\frac{k+1}{2}}$$

Conjectures

While the previous theorem established that for $k = 2$

$$d(\mathbf{P}, n) > (\log n)(\log \log n)^{\frac{1}{8}-\varepsilon}$$

which is probably not the best possible lower bound. It is currently conjectured that in k dimensions it is the case that

$$d(\mathbf{P}, n) > c_k(\log n)^k$$

In [Rot54] it was proven that $d(\mathbf{P}, n) > c_k(\log n)^{\frac{k}{2}}$. This result was improved in [BLV07] to state that there exists function $\eta(k) > 0$ such that $d(\mathbf{P}, n) > c_k(\log n)^{\frac{k}{2}+\eta(k)}$. This slow progress prompted some to speculate that the actual conjecture should be weaker, i.e.

$$d(\mathbf{P}, n) > c_k(\log n)^{\frac{k+1}{2}}$$

Both of theses conjectures match up with [Sch72] result for $k = 1$.

Yet another point of view

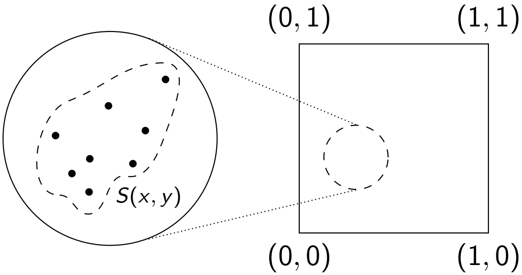
When we generalized our problem for higher dimensions we have assumed that we use k-dimensional rectangular cuboids ($[0, x) \times [0, y) \times [0, z) \times \ldots$) to measure similarity of point density.

Yet another point of view

When we generalized our problem for higher dimensions we have assumed that we use k-dimensional rectangular cuboids ($[0, x) \times [0, y) \times [0, z) \times \ldots$) to measure similarity of point density. There is however another natural approach to generalizing segment $[x, y)$ into areas, namely k-dimensional balls.

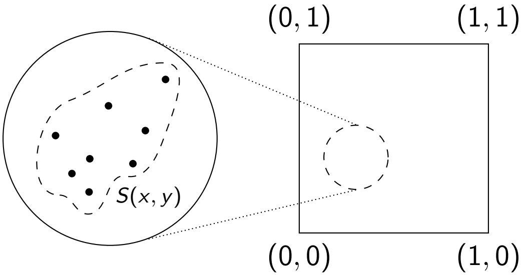
Yet another point of view

When we generalized our problem for higher dimensions we have assumed that we use k-dimensional rectangular cuboids ($[0, x) \times [0, y) \times [0, z) \times \ldots$) to measure similarity of point density. There is however another natural approach to generalizing segment $[x, y)$ into areas, namely k-dimensional balls.



Yet another point of view

When we generalized our problem for higher dimensions we have assumed that we use k-dimensional rectangular cuboids ($[0,x) \times [0,y) \times [0,z) \times \ldots$) to measure similarity of point density. There is however another natural approach to generalizing segment $[x,y)$ into areas, namely k-dimensional balls.



It turns out that this problem is much harder than the previous.

Yet another point of view continued

The only known bound is a rather weak one [Hal81]

$$d(\mathbf{P}, n) > c(\log n)^{\frac{1}{2}}$$

Yet another point of view continued

The only known bound is a rather weak one [Hal81]

$$d(\mathbf{P}, n) > c(\log n)^{\frac{1}{2}}$$

It is conjectured that in k-dimensional case

$$d(\mathbf{P}, n) > c_k(\log n)^{\frac{1}{2}}$$

but since only [Hal81] is known we do not even know whether the exponent increases with the dimension [Bil11].

Yet another point of view continued

The only known bound is a rather weak one [Hal81]

$$d(\mathbf{P}, n) > c(\log n)^{\frac{1}{2}}$$

It is conjectured that in k-dimensional case

$$d(\mathbf{P}, n) > c_k(\log n)^{\frac{1}{2}}$$

but since only [Hal81] is known we do not even know whether the exponent increases with the dimension [Bil11].
Since both balls and rectangles were considered it is natural to consider them endpoints of L_p metric for 1 and ∞ respectively and check what happens between theses extremes.

Yet another point of view continued

The only known bound is a rather weak one [Hal81]

$$d(\mathbf{P}, n) > c(\log n)^{\frac{1}{2}}$$

It is conjectured that in k-dimensional case

$$d(\mathbf{P}, n) > c_k(\log n)^{\frac{1}{2}}$$

but since only [Hal81] is known we do not even know whether the exponent increases with the dimension [Bil11].
Since both balls and rectangles were considered it is natural to consider them endpoints of L_p metric for 1 and ∞ respectively and check what happens between theses extremes. This turns out to be much simpler since it was proven that for any dimension $k \geqslant 2$ and any $p \in (1, \infty)$ it holds that

$$d(\mathbf{P}, n) > c_{k,p}(\log n)^{\frac{d}{2}}$$

Moreover this bound is sharp as there are known sequences that are of this order [Bil11].

Finishing remarks

This presentation started with considering problem of ordering a list so that it is evenly distributed. While we have tried to generalize it into more dimensions there are other approaches [Kat84]:

Finishing remarks

This presentation started with considering problem of ordering a list so that it is evenly distributed. While we have tried to generalize it into more dimensions there are other approaches [Kat84]:

- We may consider what happens if we change the required distribution from uniform to another one. Depending on the distribution can this make the problem easier or harder?

Finishing remarks

This presentation started with considering problem of ordering a list so that it is evenly distributed. While we have tried to generalize it into more dimensions there are other approaches [Kat84]:

- We may consider what happens if we change the required distribution from uniform to another one. Depending on the distribution can this make the problem easier or harder?
- In order to consider higher dimensions we had to change the problem to consider any two intervals instead of some predefined ones. Is there a predefined partitioning of higher dimension cubes into intervals that would allow us to generlize initial problem directly?

Finishing remarks

This presentation started with considering problem of ordering a list so that it is evenly distributed. While we have tried to generalize it into more dimensions there are other approaches [Kat84]:

- We may consider what happens if we change the required distribution from uniform to another one. Depending on the distribution can this make the problem easier or harder?
- In order to consider higher dimensions we had to change the problem to consider any two intervals instead of some predefined ones. Is there a predefined partitioning of higher dimension cubes into intervals that would allow us to generlize initial problem directly?
- The multidimensional problem considered L_p metrics. Could other, more exotic metrics, also be used and how would that affect bounds?

Fin

References I

[Ler04] Matyáš Lerch. “Question 1547”. In: *l'Intermédiaire des Mathématiciens* 11 (1904), pp. 145–146.

[Aar45] T van Aardenne-Ehrenfest. “Proof of the impossibility of a just distribution of an infinite sequence of points over an interval”. In: *Proc. Kon. Ned. Akad. v. Wetensch* 48 (1945), pp. 266–271.

[Aar49] T van Aardenne-Ehrenfest. “On the impossibility of a just distribution”. In: *Proc. Kon. Ned. Akad. v. Wetensch* 52 (1949), pp. 734–739.

[Rot54] K. F. Roth. “On irregularities of distribution”. In: *Mathematika* 1 (1954), pp. 73–79.

[BG70] E.R Berlekamp and R.L Graham. “Irregularities in the distributions of finite sequences”. In: *Journal of Number Theory* 2.2 (1970), pp. 152–161. ISSN: 0022-314X. DOI: https://doi.org/10.1016/0022-314X(70)90015-6. URL: https://www.sciencedirect.com/science/article/pii/0022314X70900156.

References II

[Sch72] Wolfgang Schmidt. "Irregularities of distribution, VII". In: *Acta Arithmetica* 21 (1972), pp. 45–50. DOI: 10.4064/aa-21-1-45-50. URL: https://doi.org/10.4064/aa-21-1-45-50.

[War76] M. Warmus. "A supplementary note on the irregularities of distributions". In: *Journal of Number Theory* 8.3 (1976), pp. 260–263. ISSN: 0022-314X. DOI: https://doi.org/10.1016/0022-314X(76)90002-0. URL: https://www.sciencedirect.com/science/article/pii/0022314X76900020.

[Hal81] Gabor Halasz. "On Roth's method in the theory of irregularities of point distributions". In: *Recent progress in analytic number theory* 2 (1981), pp. 79–94.

[Kat84] Amoz Kats. "Can a Party Represent its Constituency?" In: *Public Choice* 44.3 (1984), pp. 453–456. ISSN: 00485829, 15737101. URL: http://www.jstor.org/stable/30023969.

References III

[Bec89] József Beck. "A two-dimensional van Aardenne-Ehrenfest theorem in irregularities of distribution". eng. In: *Compositio Mathematica* 72.3 (1989), pp. 269–339. URL: http://eudml.org/doc/89992.

[BLV07] Dmitry Bilyk, Michael Lacey, and Armen Vagharshakyan. *On the Small Ball Inequality in All Dimensions*. 2007. DOI: 10.48550/ARXIV.0705.4619. URL: https://arxiv.org/abs/0705.4619.

[Bil11] Dmitriy Bilyk. "On Roth's orthogonal function method in discrepancy theory". In: *Uniform Distribution Theory* 1 (Jan. 2011).

[Gra13] R. L. Graham. "A note on irregularities of distribution". English. In: *Integers* 13 (2013), paper a53, 4. ISSN: 1553-1732. URL: http://math.colgate.edu/~integers/n53/n53.pdf.

References IV

[Lev20] Karl Levy. "Lower and upper bounds on irregularities of distribution". English. In: *Integers* 20 (2020), paper a26, 17. ISSN: 1553-1732. URL: http://math.colgate.edu/~integers/u26/u26.pdf.

[Kon21] S. V. Konyagin. "On Irregularity of Finite Sequences". In: *Proceedings of the Steklov Institute of Mathematics* 314.1 (Sept. 2021), pp. 90–95. DOI: 10.1134/s0081543821040052. URL: https://doi.org/10.1134/s0081543821040052.

[Sil] Tomás Oliveira e Silva. *A Problem Related to the 17 Point Problem of Steinhaus.* MathOverflow. URL:https://mathoverflow.net/q/260116 (version: 2017-01-20). eprint: https://mathoverflow.net/q/260116. URL: https://mathoverflow.net/q/260116.