

# Exponentially many 3-colorings of planar graphs with no short separating cycles

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Based on Carsten Thomassen's article [Tho21]

5 May 2022

Conjecture [Thomassen 2007]

Every planar triangle-free graph  $G$  has exponentially many 3-colorings.

## Main Theorem 1.

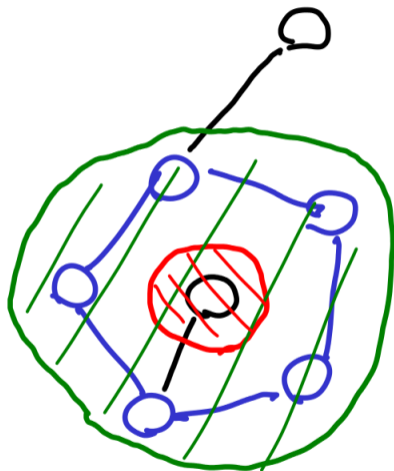
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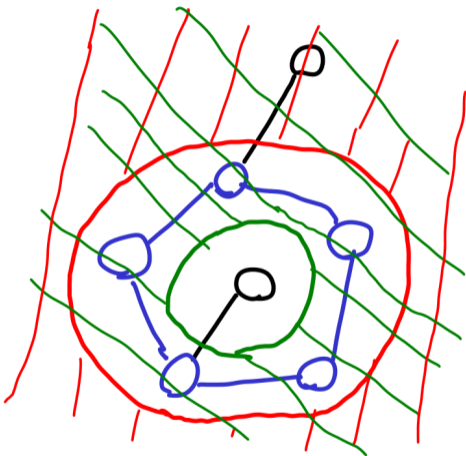
The number of proper vertex-3-colorings of every triangle-free planar graph with  $n$  vertices and with no separating cycle of length 4 or 5 is at least  $2^{n/17700000}$ .



$C$ -cycle

$\text{int}(C)$

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$ext(c)$

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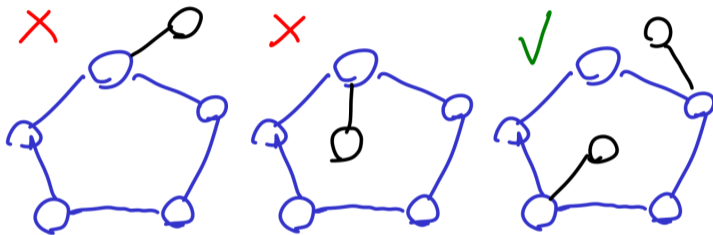


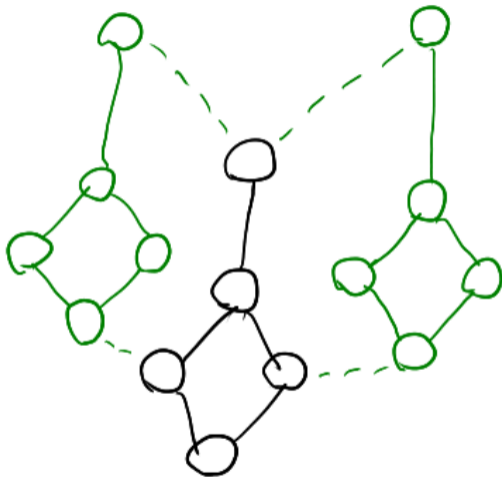
Figure: Separating cycle

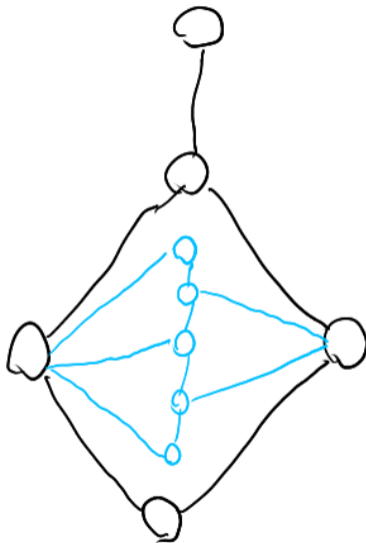
## Definition

If  $\text{int}(C)$  is empty or  $\text{ext}(C)$  is empty, then  $C$  is facial.



# Planar triangle-free graphs with only subexponentially many 3-colorings





## Theorem 3.

Let  $G$  be a connected plane graph of girth at least 4 with an outer cycle  $C$  of length 4 or 5. Let  $q$  denote the number of facial 5-cycles inside  $C$ . Let  $c$  be a 3-coloring of  $C$ .

If  $C$  has length 4, then the number of distinct 3-colorings of  $G$  extending  $c$  is at least  $2^{3q/20000}$ .

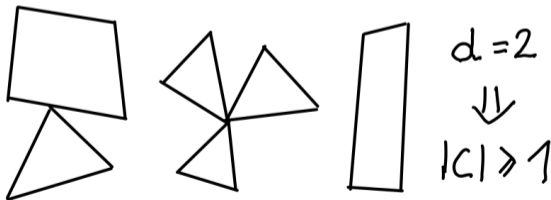
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## Theorem 4.

Let  $d$  be a natural number  $\geq 2$ , and let  $\mathcal{C}$  be a collection of  $m$  simple closed curves in the Euclidean plane such that

- 1 the interior of a curve in  $\mathcal{C}$  contains no point of any curve in  $\mathcal{C}$ ,
- 2 any two curves in  $\mathcal{C}$  have at most one point common, and
- 3 any curve in  $\mathcal{C}$  has at most  $d$  points of intersection with other curves in  $\mathcal{C}$ .

Then  $\mathcal{C}$  contains a subset  $\mathcal{C}'$  of at least  $2^{3-2d}3^{1-d}m$  curves such that each connected component  $\mathcal{C}'$  of the union of curves in  $\mathcal{C}'$  contains a point which belongs to all curves in  $\mathcal{C}'$ .

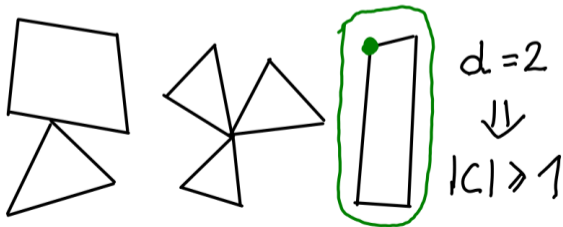


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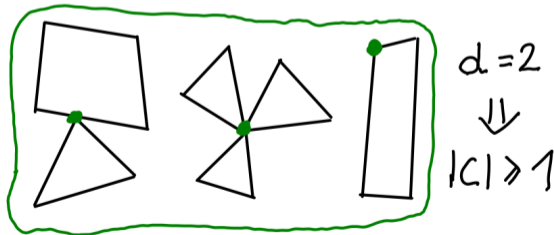


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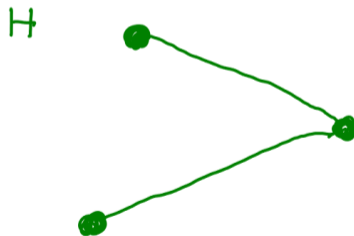
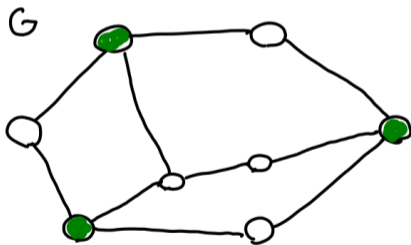
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## Theorem 5.

Let  $G$  be a graph of girth at least 4, and let  $A$  be a set of  $n$  pairwise nonadjacent vertices on the outer face boundary. Let  $H$  denote the graph with vertex set  $A$  such that two vertices in  $H$  are neighbors if and only if they are joined by a path of length (precisely) 3 in  $G$ . Then  $H$  is 16-colorable and has therefore an independent set of at least  $n/16$  vertices.



## Main Theorem 2.

The number of proper vertex-3-colorings of every triangle-free planar graph with  $n$  vertices and with no separating cycle of length 4 or 5 is at least  $2^{n/17700000}$ .

## Theorem 6.

Let  $G$  be a connected plane graph of girth at least 4 with an outer cycle  $C$  of length 4 or 5. Let  $n$  denote the number of vertices inside  $C$ . Assume  $G$  has no separating cycle of length 4 or 5. Assume also that no vertex inside  $C$  is joined to precisely two vertices on  $C$ . Let  $c$  be a 3-coloring of  $C$ . Then  $G$  has at least  $2^{n/17700000}$  distinct 3-colorings extending  $c$ .



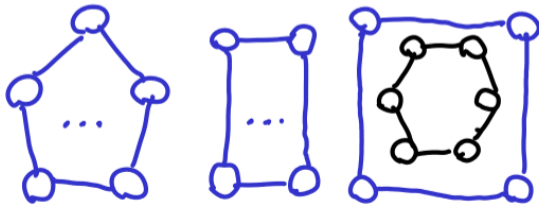
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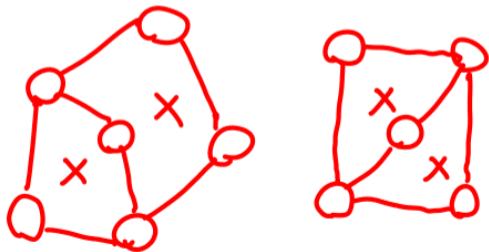
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Assume (reductio ad absurdum) that  $G$  is a smallest counterexample for  $n > 1$ .

(1): The number of 5-cycles in  $G$  is  $< n/100$ .

Proof of (1): All 5-cycles in  $G$  are facial. If the number of 5-cycles in  $G$  is at least  $n/100$ , then Theorem 3. implies that  $G$  is not a counterexample to Theorem 6, a contradiction.

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(2): The number of vertices in  $\text{int}(C)$  which have degree at most 4 and which are not contained in any 5-cycle is at least  $n/20$ .

Proof of (2): As  $G$  has no triangle,  $G$  has  $n + 4$  or  $n + 5$  vertices and at most  $2(n + 5) - 4$  edges, by Euler's formula ( $F + V - 2 = E$ ).

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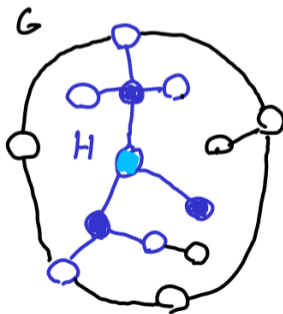
So, the number described in (2) is at least  $n - 9n/10 - n/20 = n/20$ .

(3):  $G$  contains an induced subgraph  $H$  such that each component  $H_i$  of  $H$  is a tree containing a vertex  $v$  (which we call the root of  $H_i$ ) such that

(i) each  $H_i$ -neighbor  $x$  of  $v$  is in  $\text{int}(C)$  and satisfies  $d_G(x) = d_{H_i}(x) - 4$  and is not contained in a 5-cycle, and

(ii) each vertex in  $H_i$  has distance (in both  $G$  and  $H_i$ ) at most 2 to  $v$ .

Moreover, at least  $n/1105920$  vertices  $x$  in  $H$  satisfy  $d_G(x) = d_H(x) - 4$  and are not contained in a 5-cycle.



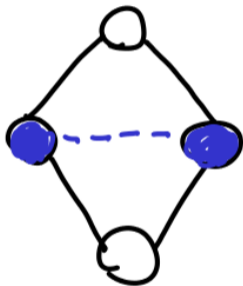
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Proof of (3): By (2) and the 4-Color Theorem,  $G$  contains an independent set  $I_1$  of at least  $n/80$  vertices, all in  $\text{int}(C)$ , all of degree at most 4, and all not contained in any 5-cycle.

$I_1$  contains a subset  $I_2$  with at least  $n/320$  vertices such that no two are in a common 4-cycle. (4-Color)

We apply Theorem 4. with  $d = 4$  to the family of curves around neighborhood of  $I_2$  vertices and let  $G$  be the subgraph of  $G$  consisting of the edges and vertices which are in the resulting subfamily. Then  $G$  contains at least  $n/276480$  vertices  $x$  that satisfy  $d_G(x) = d_G(x) - 4$  and are not contained in a 5-cycle.

$G$  may be not induced. Then some edges with ends in  $G$  connect components of  $G$ . Using the 4-Color Theorem we find a subgraph  $H$  of  $G$  consisting of pairwise nonadjacent components of  $G$  such that  $H$  contains at least  $n/1105920$  vertices  $x$  in  $H$  that satisfy  $d_G(x) = d_H(x) - 4$  and are not contained in a 5-cycle.



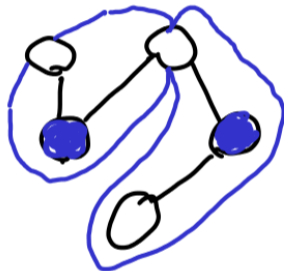
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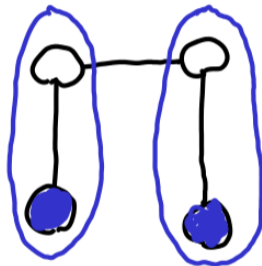
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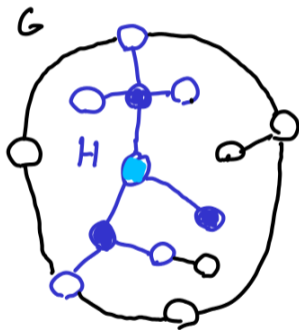
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Proof of (4):  $H$  could work, but triangle may occur after contraction.

Such triangle have only one contracted vertex ( $H$  is induced).

Such triangle corresponds to path  $P$  of length 3 in  $G$ .

$P$  must join two vertices  $y_1, y_2$  which are neighbors of distinct vertices  $x_1, x_2$  which are both neighbors of the root  $v$ .

We delete  $v$ , and for each neighbor  $x$  of  $v$  we contract  $x$  and all (at most 3) neighbors of  $x$  in  $G \setminus v$  into a single vertex.

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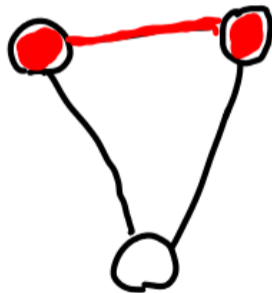
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Theorem 5 on the graph with  $A$  being the set of contracted vertices results in a subset  $A^0$  of  $A$  with at least  $|A^0| = 16$  vertices. Each vertex in  $A$  corresponds to a vertex  $x$  (joined to a root  $v$  that was deleted).  $Q$  is the subgraph induced by  $x$ 's with neighborhood.

(4): The subgraph  $H$  defined in (3) contains an induced subgraph  $Q$  with at least  $n=17694720$  vertices  $x$  satisfying  $d_G(x) = d_Q(x) - 4$  and  $x$  not contained in a 5-cycle, such that the contraction of each component  $Q^0$  of  $Q$  into a vertex  $q$  results in a triangle-free planar graph.

Proof of (4):  $H$  could work, but triangle may occur after contraction.

Such triangle have only one contracted vertex  $x$  (is induced).

Such triangle corresponds to path  $P$  of length 3 in  $G$ .

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We contract each component  $Q^0$  of  $Q$  (defined in (4)) into a vertex  $q$ .  
By Grötzsch's theorem, the resulting graph has a 3-coloring extending the coloring of  $C$ .

We color roots and grandchildren in  $Q^0$  components to the color of  $q$ .

Now, there are at least  $n=17694720$  uncolored vertices in  $G$  with same color neighbors.

There are at least  $2^{n=17694720}$  color extensions of the coloring of  $C$ .  
Contradiction,  $G$  was not a counterexample.

Theorem 9

Let  $G$  be a connected plane graph of girth at least 4 with an outer cycle  $C$  of length 4 or 5. Let  $n$  denote the number of vertices inside  $C$ . Assume  $G$  has no separating cycle of length 4 or 5. Assume also that no vertex inside  $C$  is joined to precisely two vertices on  $C$ . Let  $c$  be a 3-coloring of  $C$ . Then  $G$  has at least  $2^{n=17700000}$  distinct 3-colorings extending  $c$ .

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Theorem 2

Let  $G$  be a connected plane graph of girth at least 4 with an outer cycle  $C$  of length 4 or 5. Let  $n$  denote the number of vertices inside  $C$ . Assume  $G$  has no separating cycle of length 4 or 5. Assume also that no vertex inside  $C$  is joined to precisely two vertices on  $C$ . Let  $c$  be a 3-coloring of  $C$ . Then  $G$  has at least  $2^{n=17700000}$  distinct 3-colorings extending  $c$ .

$$20 = 20$$

Degree 4 and no 5-cycle

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$$20 \cdot 4 = 80$$

Degree 4 and no 5-cycle

Independent set

$$20 = 20$$

$$20 \cdot 4 = 80$$

$$20 \cdot 4 \cdot 4 = 320$$

Degree 4 and no 5-cycle

Independent set

No common 4-cycle

$$20 = 20$$

$$20 \cdot 4 = 80$$

$$20 \cdot 4 \cdot 4 = 320$$

$$20 \cdot 4 \cdot 4 \cdot 864 = 276480$$

Degree 4 and no 5-cycle

Independent set

No common 4-cycle

Snow flakes

$$20 = 20$$

$$20 \cdot 4 = 80$$

$$20 \cdot 4 \cdot 4 = 320$$

$$20 \cdot 4 \cdot 4 \cdot 864 = 276480$$

$$20 \cdot 4 \cdot 4 \cdot 864 \cdot 4 = 1105920$$

Degree 4 and no 5-cycle

Independent set

No common 4-cycle

Snowflakes

Subgraph H

$$20 = 20$$

$$20 \cdot 4 = 80$$

$$20 \cdot 4 \cdot 4 = 320$$

$$20 \cdot 4 \cdot 4 \cdot 864 = 276480$$

$$20 \cdot 4 \cdot 4 \cdot 864 \cdot 4 = 1105920$$

$$20 \cdot 4 \cdot 4 \cdot 864 \cdot 4 \cdot 16 = 17694720$$

Degree 4 and no 5-cycle

Independent set

No common 4-cycle

Snow akes

SubgraphH

SubgraphQ



- [Tho21] Carsten Thomassen. “Exponentially many 3-colorings of planar triangle-free graphs with no short separating cycles”. In: *Journal of Combinatorial Theory, Series B* (2021). ISSN: 0095-8956. DOI: <https://doi.org/10.1016/j.jctb.2021.01.009>. URL: <https://www.sciencedirect.com/science/article/pii/S0095895621000095>.