Contact Graphs on Ball Packings

Szymon Salabura

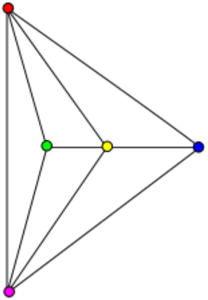
5th May 2022

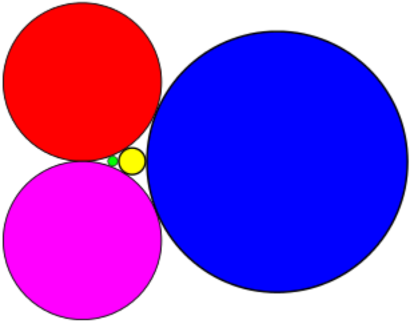
Table of Contents

Introduction

A packing of balls in $\mathbb{R}^d$ is a finite set of balls with non-intersecting interiors.

Each packing naturally entails a contact graph where graph vertices are the balls of the packing and two vertices are connected by an edge if and only if the corresponding balls are tangent.

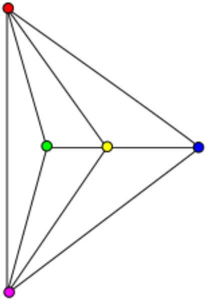


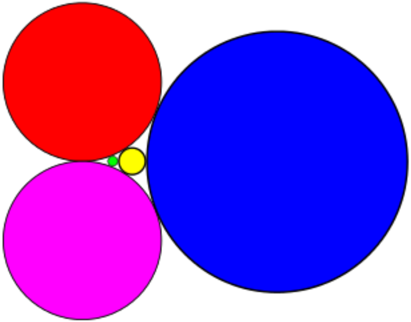


Introduction

For each contact graph G of a closed ball packing in $\mathbb{R}^d$ denote its average degree by $k(G)$.

Define $k_d = \sup k(G)$ taken over all contact graphs.



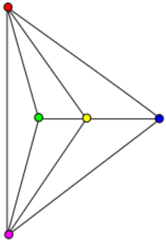


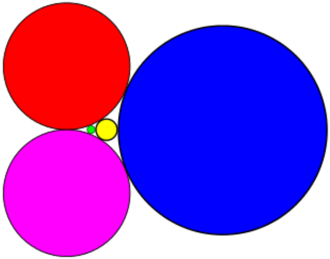
Introduction

The problem of characterizing planar disk packings is completely solved by the following theorem:

Koebe-Andreev-Thurston Theorem.

For every simple planar graph G there is a set of non-intersecting closed disks on the plane whose contact graph is G.



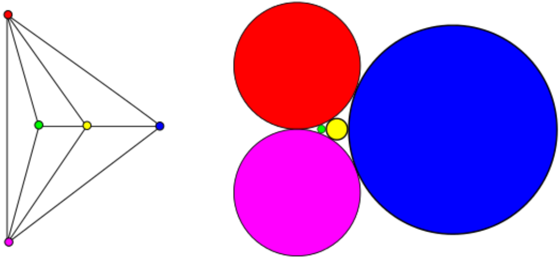


Introduction

The problem of characterizing planar disk packings is completely solved by the following theorem:

Koebe-Andreev-Thurston Theorem.

For every simple planar graph G there is a set of non-intersecting closed disks on the plane whose contact graph is G.



In particular, $k_2 < 6$.

A simple way to bound k_d is by using kissing numbers.

Definition.

By a kissing number τ_d we mean the maximum number of non-overlapping closed unit balls tangent to a given unit ball in $\mathbb{R}^d$.

Introduction

A simple way to bound k_d is by using kissing numbers.

Definition.

By a kissing number τ_d we mean the maximum number of non-overlapping closed unit balls tangent to a given unit ball in $\mathbb{R}^d$.

In one dimension, $\tau_1 = 2$:

Introduction

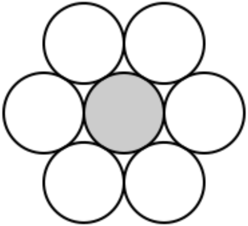
A simple way to bound k_d is by using kissing numbers.

Definition.

By a kissing number τ_d we mean the maximum number of non-overlapping closed unit balls tangent to a given unit ball in $\mathbb{R}^d$.

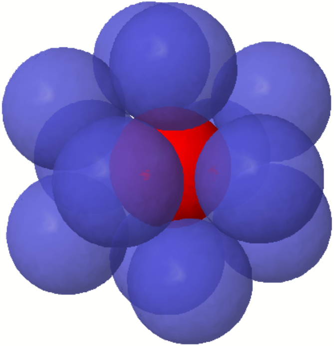
In one dimension, $\tau_1 = 2$:

In two dimensions, $\tau_2 = 6$:



In three dimensions, $\tau_3 = 12$, but the correct value was much more difficult to establish than in dimensions one and two.

It is easy to arrange 12 spheres so that each touches a central sphere, but there is a lot of space left over, and it is not obvious that there is no way to pack in a 13^{th} sphere.



Introduction

It is easy to show that $k_d \leqslant 2\tau_d$, since each ball cannot have more than τ_d larger or equal balls tangent to it.

Introduction

It is easy to show that $k_d \leqslant 2\tau_d$, since each ball cannot have more than τ_d larger or equal balls tangent to it.

The bounds for τ_d imply the following bounds for k_d:

- $k_3 \leqslant 24$,
- $k_4 \leqslant 48$,
- $k_5 \leqslant 88$,
- $k_6 \leqslant 156$,
- $k_7 \leqslant 268$,
- $k_8 \leqslant 480$,
- and so on...

Introduction

Kuperberg-Schramm (1994)

$$12.56 \approx 666/53 \leqslant k_3 < 8 + 4\sqrt{3} \approx 14.93$$

Introduction

Kuperberg-Schramm (1994)

$$12.56 \approx 666/53 \leqslant k_3 < 8 + 4\sqrt{3} \approx 14.93$$

Eppstein-Kuperberg-Ziegler (2003)

$$12.61 \approx 7656/607 \leqslant k_3$$

Introduction

Kuperberg-Schramm (1994)

$$12.56 \approx 666/53 \leqslant k_3 < 8 + 4\sqrt{3} \approx 14.93$$

Eppstein-Kuperberg-Ziegler (2003)

$$12.61 \approx 7656/607 \leqslant k_3$$

The main results of the paper are the new upper bounds:

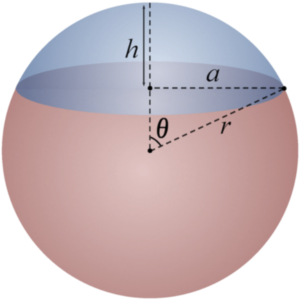
$$k_3 < 13.92, k_4 < 34.69, k_5 < 77.76$$

Table of Contents

Kuperberg-Schramm approach

Area of spherical cap

$$A = 2\pi rh$$



Kuperberg-Schramm approach

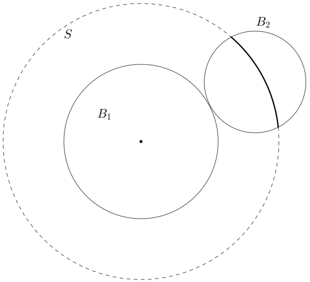
Proposition.

$$\tau_3 \leqslant 8 + 4\sqrt{3}$$

Proof.

For a unit ball B_1 in $\mathbb{R}^3$, consider a concentric sphere S with radius $\sqrt{3}$. Any unit sphere tangent to B_1 intersects this sphere by a spherical cap with height $\sqrt{3} - 3/2$.

The area of this cap is $2\pi\sqrt{3}(\sqrt{3} - 3/2) = (6 - 3\sqrt{3})\pi$. Since the area of the sphere is 12π, no more than $\frac{12\pi}{(6-3\sqrt{3})\pi} = 8 + 4\sqrt{3}$ spherical caps may fit in its surface. □



Kuperberg-Schramm approach

The same idea of bounding the number of tangent spheres is not directly applicable when different radii are allowed. However, for two tangent balls the smaller proportion of area taken by a smaller ball on a sphere concentric to a larger ball is compensated by a larger proportion of area taken by a larger ball on a sphere concentric to a smaller ball.

This is the foundation of the approach by Kuperberg and Schramm.

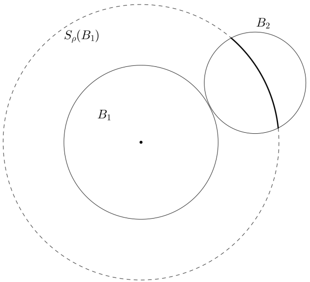
Kuperberg-Schramm (1994)

$$k_3 < 8 + 4\sqrt{3} \approx 14.93$$

Kuperberg-Schramm approach

Fix $\rho > 1$. For each ball B denote the concentric sphere with radius ρ times larger by $S_\rho(B)$. For two tangent balls B_1 and B_2 define

$$a(B_1, B_2) = \frac{\text{area}(S_\rho(B_1) \cap B_2)}{\text{area}(S_\rho(B_1))}.$$



Kuperberg-Schramm approach

Remarkably, if both $S_\rho(B_1) \cap B_2$ and $S_\rho(B_2) \cap B_1$ are non-empty, $a(B_1, B_2) + a(B_2, B_1)$ depends only on ρ.

From this moment on, we consider only $\rho < 3$; otherwise at least one of $a(B_1, B_2)$ and $a(B_2, B_1)$ is 0.

Lemma.

$$a(B_1, B_2) + a(B_2, B_1) = \frac{1}{2}\left(\frac{h_1}{\rho r_1} + \frac{h_2}{\rho r_2}\right)$$

Kuperberg-Schramm approach

Remarkably, if both $S_\rho(B_1) \cap B_2$ and $S_\rho(B_2) \cap B_1$ are non-empty, $a(B_1, B_2) + a(B_2, B_1)$ depends only on ρ.

From this moment on, we consider only $\rho < 3$; otherwise at least one of $a(B_1, B_2)$ and $a(B_2, B_1)$ is 0.

Lemma.

$$a(B_1, B_2) + a(B_2, B_1) = \frac{1}{2}\left(\frac{h_1}{\rho r_1} + \frac{h_2}{\rho r_2}\right)$$

Lemma.

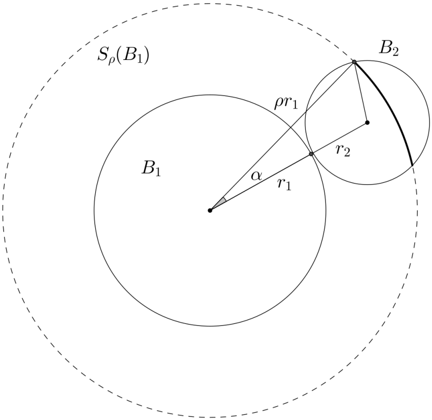
$$\frac{h_1}{\rho r_1} + \frac{h_2}{\rho r_2} = \frac{-\rho^2 + 4\rho - 3}{2\rho}$$

if both $S_\rho(B_1) \cap B_2$ and $S_\rho(B_2) \cap B_1$ are non-empty, otherwise left hand size is greater.

Kuperberg-Schramm approach

Proof.

$\frac{h_1}{\rho r_1} = 1 - \cos\alpha$, where α is the spherical radius of the cap $S_\rho(B_1) \cap B_2$.

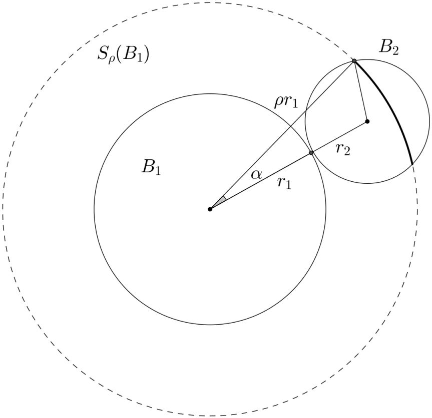


Kuperberg-Schramm approach

By law of cosines:

$$(\rho r_1)^2 + (r_1 + r_2)^2 - 2\rho r_1(r_1 + r_2)\cos\alpha = r_2^2$$

$$\cos\alpha = \frac{(\rho^2 + 1)r_1 + 2r_2}{2\rho(r_1 + r_2)}$$

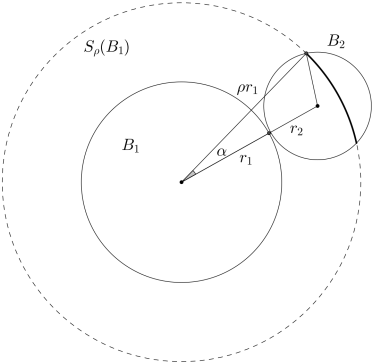


Kuperberg-Schramm approach

Similarly for the radius β of the second cap. Therefore, we get:

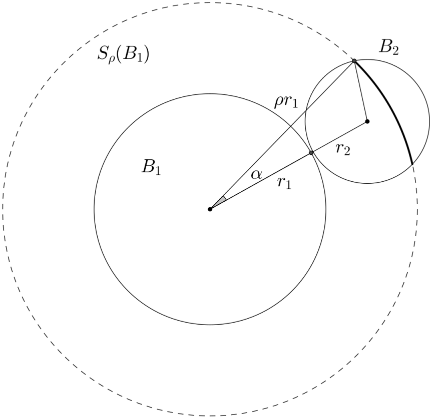
$$\cos\alpha + \cos\beta = \frac{(\rho^2+1)r_1 + 2r_2}{2\rho(r_1+r_2)} + \frac{(\rho^2+1)r_2 + 2r_1}{2\rho(r_1+r_2)} = \frac{\rho^2+3}{2\rho}$$

$$\frac{h_1}{\rho r_1} + \frac{h_2}{\rho r_2} = 2 - (\cos\alpha + \cos\beta) = \frac{-\rho^2+4\rho-3}{2\rho}$$



Kuperberg-Schramm approach

If e.g. $S_\rho(B_1) \cap B_2$ is empty ($h_1{=}0$), then $\frac{h_1}{\rho r_1} = 0$. Increasing r_1, we only increase the second term $\frac{h_2}{\rho r_2}$. Therefore, in the case when one intersection is empty the inequality holds. $\square$



Kuperberg-Schramm approach

Denote by $\mathrm{dens}(\rho)$ the supremum of $\sum_i a(B, B_i)$ taken over all balls in the packing, where B_i are all balls tangent to B.

In a contact graph $G = (V, E)$ we get:

$$\sum_{\{X,Y\}\in E} (a(X,Y) + a(Y,X)) \geqslant \frac{-\rho^2 + 4\rho - 3}{4\rho} |E|$$

$$\sum_{\{X,Y\}\in E} (a(X,Y) + a(Y,X)) \leqslant \mathrm{dens}(\rho)|V|$$

$$2|E|/|V| \leqslant \frac{8\rho}{-\rho^2 + 4\rho - 3} \mathrm{dens}(\rho)$$

Kuperberg-Schramm approach

Theorem.

$$k_3 \leqslant \inf_{1<\rho<3} \left\{ \frac{8\rho}{-\rho^2 + 4\rho - 3} \mathrm{dens}(\rho) \right\}$$

Kuperberg and Schramm used $\mathrm{dens}(\rho) \leqslant 1$ and, taking the optimum $\rho = \sqrt{3}$, proved their upper bound $k_3 \leqslant 8 + 4\sqrt{3}$.

Table of Contents

Bounds in higher dimensions

Similar approach works in higher dimensions.

$(d-1)$-dimensional area of a spherical cap with spherical radius α on the unit sphere in $\mathbb{R}^d$

$$A = \frac{\pi^{d/2}}{\Gamma(d/2)} \int_0^{\sin^2 \alpha} t^{\frac{d-3}{2}} (1-t)^{-\frac{1}{2}} \, dt$$

For $d = 3$, this formula is equivalent to $A = 2\pi rh$.

Bounds in higher dimensions

Theorem.

$$k_d \leqslant \frac{2\int_0^1 t^{\frac{d-3}{2}}(1-t)^{-\frac{1}{2}}\,dt}{\int_0^{1/4} t^{\frac{d-3}{2}}(1-t)^{-\frac{1}{2}}\,dt}$$

Bounds in higher dimensions

Theorem.

$$k_d \leqslant \frac{2\int_0^1 t^{\frac{d-3}{2}}(1-t)^{-\frac{1}{2}}\,dt}{\int_0^{1/4} t^{\frac{d-3}{2}}(1-t)^{-\frac{1}{2}}\,dt}$$

For $d = 4, 5$, this theorem gives the new upper bounds on k_d.

$$k_4 < 34.69, k_5 < 77.76$$

Starting from 6, upper bounds based on kissing numbers become better.

Table of Contents

New bound in dimension 3

Definition.

Let $K(\alpha)$ be a non-decreasing function defined on $I = [\alpha_{min}, \alpha_{max}]$, $0 < \alpha_{min} \leqslant \alpha_{max} \leqslant \frac{\pi}{2}$. For a packing $\mathcal{C}$ of a unit sphere with circles whose radii belong to I, the density is defined as

$$d(\mathcal{C}) = \frac{1}{4\pi} \sum_{C \in \mathcal{C}} K(radius(C)).$$

New bound in dimension 3

Definition.

Let $K(\alpha)$ be a non-decreasing function defined on $I = [\alpha_{min}, \alpha_{max}]$,$0 < \alpha_{min} \leqslant \alpha_{max} \leqslant \frac{\pi}{2}$. For a packing $\mathcal{C}$ of a unit sphere with circles whose radii belong to I, the density is defined as

$$d(\mathcal{C}) = \frac{1}{4\pi} \sum_{C \in \mathcal{C}} K(radius(C)).$$

We can think of K as a weight function so $\sum K(radius(C))$ is the total weight of all spherical caps in a packing. Then $d(\mathcal{C})$ represents the total weight of the packing.

New bound in dimension 3

Definition.

For $x, y, z \in I$, we consider a spherical triangle Δ formed by centers of pairwise tangent circles of radii x, y, z. The density of this triangle is defined by

$$D(x, y, z) = \frac{1}{2\pi \cdot area(\Delta)}(K(x)\angle x + K(y)\angle y + K(z)\angle z)$$

where $\angle x, \angle y, \angle z$ are measures of spherical angles with vertices at centers of circles of radii x, y, z respectively.

New bound in dimension 3

Definition.

For $x, y, z \in I$, we consider a spherical triangle Δ formed by centers of pairwise tangent circles of radii x, y, z. The density of this triangle is defined by

$$D(x, y, z) = \frac{1}{2\pi \cdot area(\Delta)}(K(x)\angle x + K(y)\angle y + K(z)\angle z)$$

where $\angle x, \angle y, \angle z$ are measures of spherical angles with vertices at centers of circles of radii x, y, z respectively.

For a spherical angle $\angle x$, the part of the spherical cap of radius x that is inside the angle is $\frac{\angle x}{2\pi}$ so its weight is $K(x)\frac{\angle x}{2\pi}$. The total weight of the triangle may be calculated as $\frac{1}{2\pi}(K(x)\angle x + K(y)\angle y + K(z)\angle z)$.

New bound in dimension 3

Theorem.

$$d(\mathcal{C}) \leqslant \max_{x,y,z \in \boldsymbol{I}} D(x, y, z)$$

The main conclusion is that, in order to bound the maximum density, it is sufficient to consider only triangles formed by three pairwise tangent spherical caps.

New bound in dimension 3

Theorem.

$$k_3 \leqslant \inf_{1<\rho<3} \left\{ \max_{x,y,z \in I_\rho} D_\rho(x,y,z) \frac{8\rho}{-\rho^2 + 4\rho - 3} \right\}$$

New bound in dimension 3

Theorem.

$$k_3 \leqslant \inf_{1<\rho<3} \left\{ \max_{x,y,z \in I_\rho} D_\rho(x,y,z) \frac{8\rho}{-\rho^2+4\rho-3} \right\}$$

Corollary.

$$k_3 < 13.92$$

Table of Contents

New upper bounds

After the publication of the paper, Dostert, Kolpakov and Oliveira found new upper bounds for k_d in dimensions $3, \dots, 9$.

d	$2\tau_d$	Glazyrin	D-K-O
3	24	13.92	13.61
4	48	34.69	27.44
5	88	77.76	64.03
6	156	-	121.11
7	268	-	223.15
8	480	-	408.39
9	726	-	722.63