Nash-Williams Theorem

Let G be a graph with multiple edges allowed but no loops. A ρ-forest decomposition is a decomposition of the edge set into ρ subsets $E(G) = E_1 \cup \ldots E_\rho$ such that each subgraph E_i is acyclic. The minimum ρ is called the arboricity of G, and denoted by $\rho(G)$.

Nash-Williams Theorem:

$$\rho(G) = max_H \lceil e(H)/(v(H) - 1) \rceil$$

where H runs over all subgraphs of G with $v(H) := |V(H)| > 1$, and $e(H) := |E(H)|$.

Proof

Note: $\rho(G) \geq \lceil e(H)/(v(H)-1) \rceil$ is a trivial lower bound.

Proof. Let G be a counter-example that minimizes $e(G)+v(G)$. Then $\rho(G)$ is strictly greater than the right side of the equation.

Obviously G is connected with $\rho(G)$ $>$1 and *critical* with respect to the arboricity, that is, $\rho(G-e) < \rho(G)$ holds for each $e \in E$.

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Lemma 1. Let G be connected and critical with $\rho(G) > 1$. Then for every $e \in E$ any $(\rho(G)-1)$-forest decomposition of $G-e$ is a decomposition into $\rho(G)-1$ spanning trees of G.

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Lemma 1. Let G be connected and critical with $\rho(G) > 1$. Then for every $e \in E$ any $(\rho(G) - 1)$-forest decomposition of $G - e$ is a decomposition into $\rho(G) - 1$ spanning trees of G.

Using this lemma, we obtain an equality:

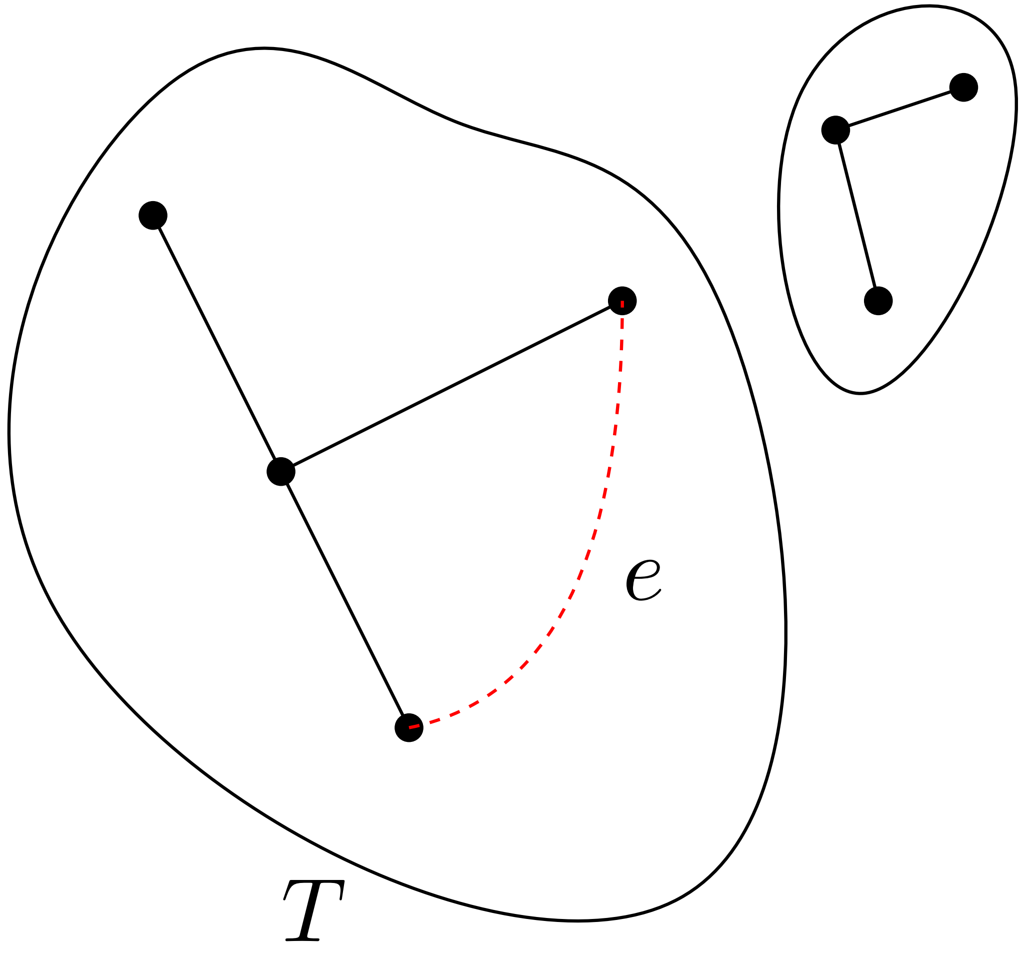
$$e(G) = (\rho(G) - 1) \cdot (v(G) - 1) + 1$$

which leads to the following contradiction:

$$\rho(G) > \lceil e(G)/(v(G){-}1) \rceil = \lceil ((\rho(G){-}1)(v(G){-}1){+}1)/(v(G){-}1) \rceil = \lceil \rho(G){-}1{+}1/(v(G){-}1) \rceil = \rho(G)$$

Proof of Lemma 1

Proof by contradiction. Let $\rho := \rho(G)$ and let $E_1 \ldots E_{\rho-1}$ be a forest decomposition of $G - e$ where E_1 is **not** a spanning tree of G. Since $E_1 + e$ must contain a cycle, both ends of e are in a connected component T of E_1.



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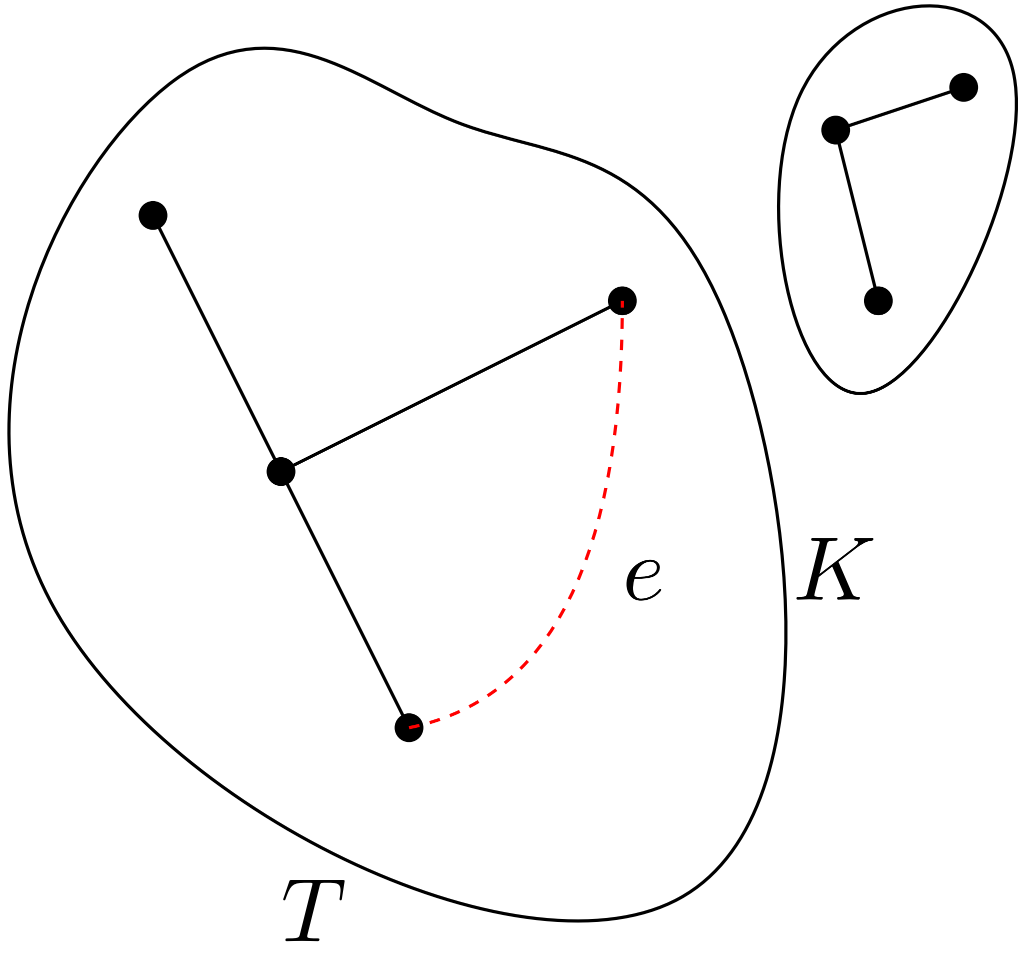

Let K be a subgraph induced by $V(T)$. $K \neq G$ and by the criticality of the the G, K has $(\rho - 1)$-forest decomposition: $E(K) = A_1 \cup \ldots \cup A_{\rho-1}$.

Let $S := \{(E'_1 \ldots, E'_{\rho-1}, \{e'\}) : \rho$-forest decomposition of G, such that a connected component of E_1 is a spanning tree of K and $e' \in K\}$.

$(E_1, \ldots E_{\rho-1}, \{e\}) \in S$ shows that $|S| > 0$. Let $(\overline{E_1} \ldots \overline{E_{\rho-1}}, \{\overline{e}\})$ be an element of S that maximizes:

$$J(\overline{E}) = \sum_{i=1}^{\rho-1} |A_i \cap \overline{E_i}|$$

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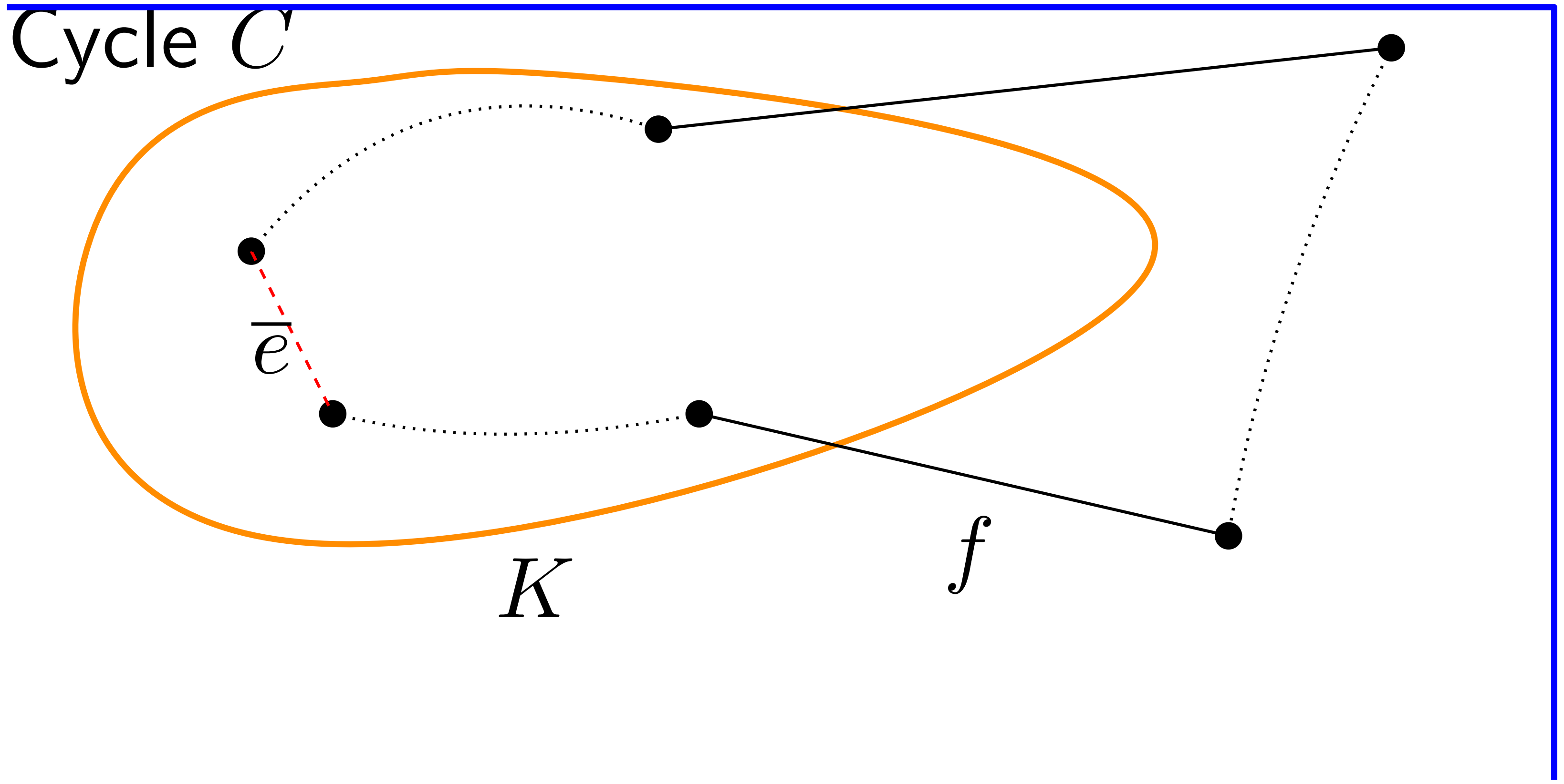
Since $\overline{e} \in E(K)$, $\overline{e} \in A_t$ for some t. $E_t + \overline{e}$ must contain a cycle C. We will prove that $C \subset K$. Case $t = 1$ is trivial. Otherwise if $C \not\subset K$ then we can take an edge $f \in C$ with one end in $V(K)$ and the other in $V(G) \setminus V(K)$. $\overline{E_1} + f$ is acyclic and then $(\overline{E_1} + f, \ldots \overline{E_t} + \overline{e} - f, \ldots, \overline{E_\rho})$ is $\rho - 1$ decomposition of G – contradiction.



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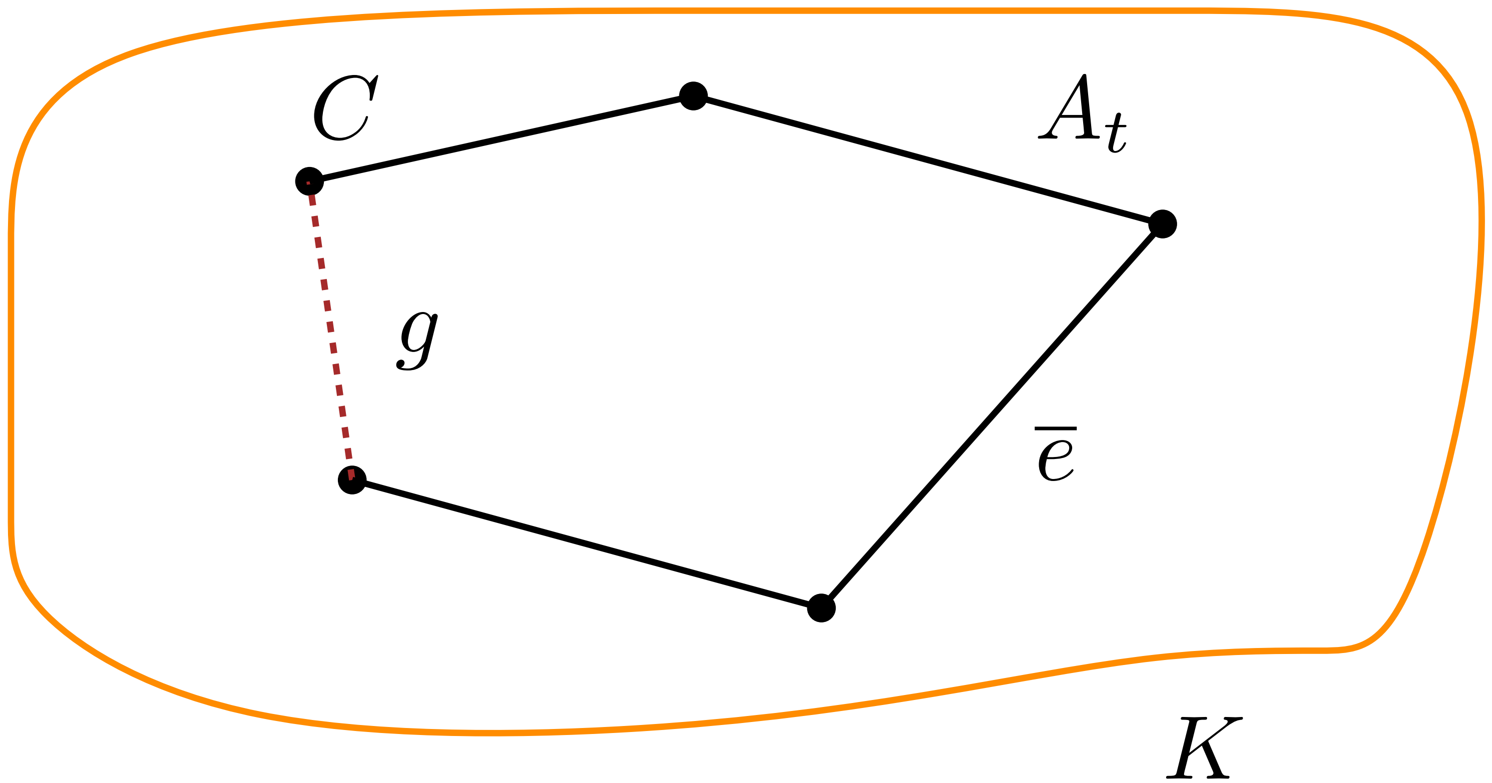

Since A_t is acyclic, there exists an edge $g \in E(C) - A_t$. Now $(\overline{E_1} \ldots \overline{E_t} + \overline{e} - g, \ldots \overline{E_{\rho-1}}, \{g\}) \in S$ increases J by one, which contradicts the assumption on maximality.

Thorem 2. (Reiher, Sauermann). Given a graph G, an integer $\rho = \rho(G)$, and moreover a sequence $e_1, e_2, ..., e_\rho$ of distinct edges of G, there exists a partition $(E_1, \ldots E_\rho)$ such that $e_i \in E_i$ and E_i's are forests.

Proof by contradiction. Let $(E_1, \ldots E_\rho)$ be a partition, that minimizes the number of i's such that $e_i \notin E_i$. If $e_k \notin E_k$ then $e_k \in E_l$ for $l \neq k$.

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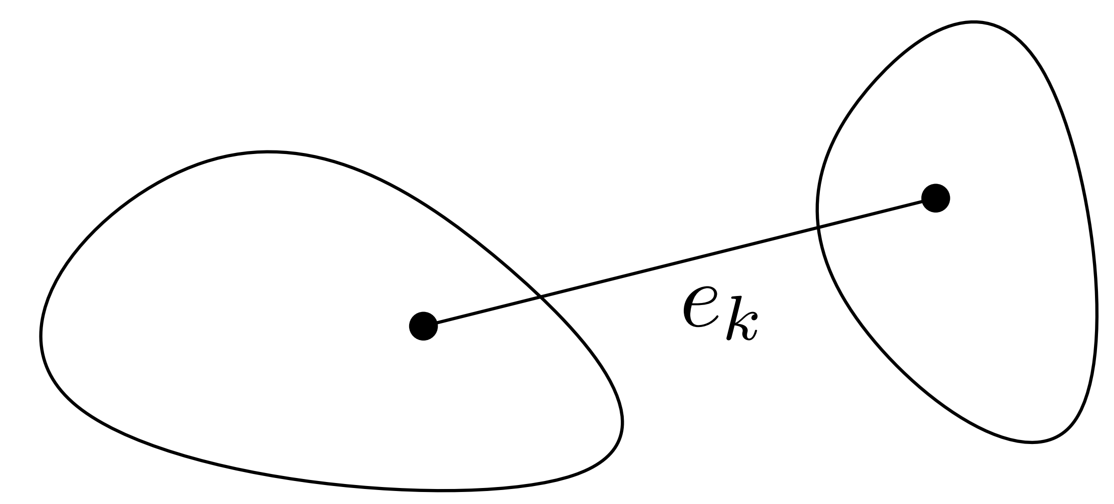

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Let C be a cycle in $E_k + e_k$. u, v are in different components of $(E_l - e_k)$. Hence, there is an edge f of the cycle C connecting vertices of different components of $(E_l - e_k)$.

The partition gained from $(E_1, \dots E_\rho)$ substituting E_k by $E_k + e_k - f$ and E_l by $E_l - e_k + f$ is better.