Bears with Hats and Independence Polynomials

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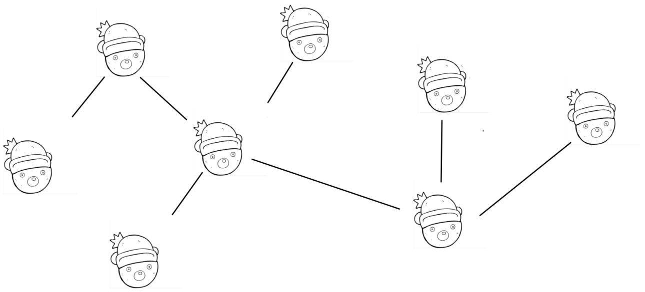
Hat guessing game



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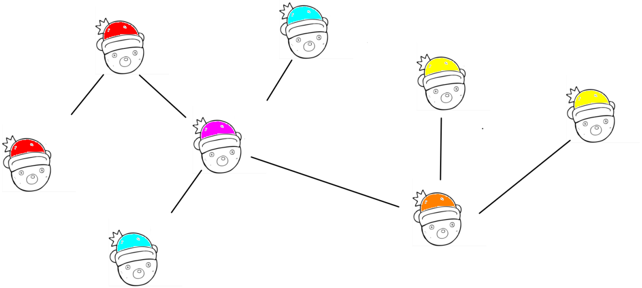
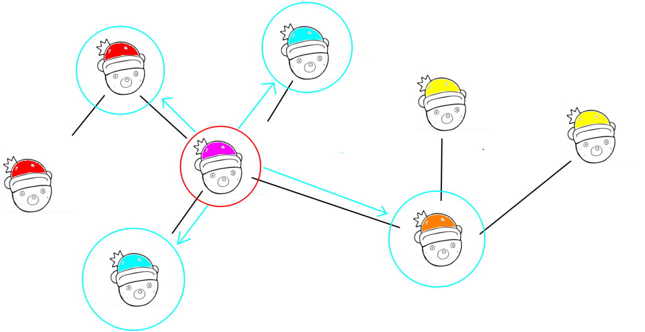


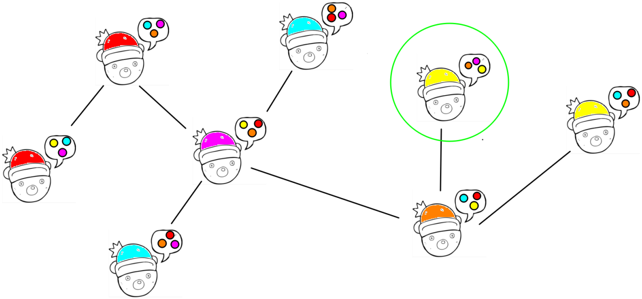
- We are given a graph G. In each vertex sits a bear.
- A demon puts colorful hats on the bears. Each hat has one of h colors.

Hat guessing game



- Bears see hats of their neighbours. Based on this information and a predetermined strategy, the bears guess the colors of their hats.

Hat guessing game



- Each bear has g tries. The bears win if at least one bear guesses correctly.

Formal definition

A **hat guessing game** is a triple $H = (G, h, g)$ where

- $G = (V, E)$ is an undirected graph.
- $h \in \mathbb{N} :=$ number of different possible hat colors for each bear
- $g \in \mathbb{N} :=$ the number of guesses each bear is allowed to make

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A strategy is **winning** if

$$\forall_{\varphi} \exists_{v} \varphi(v) \in \Gamma_v((\varphi(u))_{u \in N(v)})$$

.

Non-uniform variant

$$(G = (V, E), \mathbf{h}, \mathbf{g})$$

$$\mathbf{h} = (h_v)_{v \in V} \text{ and } \mathbf{g} = (g_v)_{v \in V}$$

.

A bear on v gets a hat of one of h_v colors and is allowed to guess exactly g_v colors.

(Fractional) Hat Chromatic Number

The **hat chromatic number** $\mu(G) := \max h$ for which game $(G, h, 1)$ is winning.

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The **fractional hat chromatic number** $\hat{\mu}(G)$ is defined as

$$\hat{\mu}(G) = \sup \left\{ \frac{h}{g} \mid (G, h, g) \text{ is a winning game} \right\}$$

Fractional hat chromatic number doesn't have to be rational (paths).

(Fractional) Hat Chromatic Number

Observation. Let $k \in \mathbb{N}$. If a game $H = (G, h, g)$ is winning, then the game $H_k = (G, k \cdot h, k \cdot g)$ is winning as well.

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Lemma. Suppose that

- (G, h, g) is winning, and
- $r' \in \mathbb{Q}$, $r' \leq h/g$

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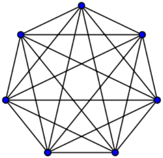
Corollary.: If $\frac{p}{q} < \hat{\mu}(G)$, then there are $h, g \in \mathbb{N}$ such that $\frac{p}{q} = \frac{h}{g}$ and the bears win the game (G, h, g).

Fractional Hat Chromatic Number - Cliques



Theorem. Bears win a game $(K_n = (V, E), \mathbf{h}, \mathbf{g})$ if and only if

$$\sum_{v \in V} \frac{g_v}{h_v} \geq 1$$

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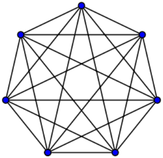

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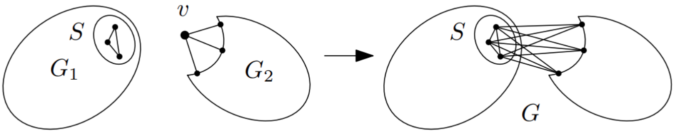
Corollary. For each $n \in \mathbb{N}$, it holds that $\hat{\mu}(K_n) = n$

Clique Join



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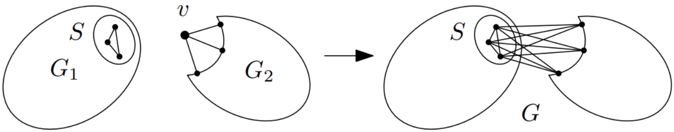

Lemma. Games

$$\mathcal{H}_1 = \left(G_1 = (V_1, E_1), \mathbf{h}^1, \mathbf{g}^1\right)$$

$$\mathcal{H}_2 = \left(G_2 = (V_2, E_2), \mathbf{h}^2, \mathbf{g}^2\right)$$

are winning $\implies$ game $\mathcal{H} = (G, \mathbf{h}, \mathbf{g})$ is winning, where

$$h_u = \begin{cases} h_u^1 & u \in V_1 \backslash S \\ h_u^2 & u \in V_2 \backslash \{v\} \\ h_u^1 \cdot h_v^2 & u \in S, \text{ and} \end{cases} \qquad g_u = \begin{cases} g_u^1 & u \in V_1 \backslash S \\ g_u^2 & u \in V_2 \backslash \{v\} \\ g_u^1 \cdot g_v^2 & u \in S. \end{cases}$$

Independence Polynomial

The **multivariate independence polynomial** of a graph $G = (V, E)$ on variables $\mathbf{x} = (x_v)_{v \in V}$ is

$$P_G(\mathbf{x}) = \sum_{\substack{I \subseteq V \\ I \text{ independent set}}} \prod_{v \in I} x_v .$$

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Inclusion-exclusion principle. For a union A of sets $A_1, \ldots, A_n$ holds that

$$|A| = \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right| .$$

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Idea: We can use the inclusion-exclusion principle to compute the probability that at least one bear sitting on some vertex of I guesses correctly, where I is an independent set.

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$\mathcal{R}(G) :=$ set of all vectors $\mathbf{r} \in [0,\infty)^V$ such that $Z_G(\mathbf{w}) > 0$ for all $0 \leq \mathbf{w} \leq \mathbf{r}$, where the comparison is done entry-wise.

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Note that $Z_G(\mathbf{0}) = 1$

Independence Polynomial

Theorem. Let $G = (V, E)$ be a graph.

- $(A_v)_{v \in V}$ is a family of events, A_v is independent of $\{A_w \mid w \notin N^+(v)\}$
- $\mathbf{p} \in [0,1]^V$, for each v we have $P(A_v) \leq p_v$, $\mathbf{p} \in \mathcal{R}(G)$

Then

$$P\left(\bigcap_{v \in V} \bar{A}_v\right) \geq Z_G(\mathbf{p}) > 0.$$

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Proposition. A hat guessing game $\mathcal{H} = (G = (V, E), \mathbf{h}, \mathbf{g})$ is losing whenever $\mathbf{r} \in \mathcal{R}(G)$ where $\mathbf{r} = (g_v / h_v)_{v \in V}$.

Perfect Strategy

A strategy $\mathcal{H}$ is **perfect** if

- It is winning, and
- In every hat arrangement, no two bears that guess correctly are on adjacent vertices

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Proposition. $(G, \mathbf{h}, \mathbf{g})$ has perfect strategy $\implies$ for $\mathbf{r} = (g_v/h_v)_{v \in V}$ we have

$$Z_G(\mathbf{r}) = 0 \text{ and } Z_G(\mathbf{w}) \geq 0$$

for every $0 \leq \mathbf{w} \leq \mathbf{r}$.

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&= m \cdot (1 - Z_G(\mathbf{r})) \implies Z_G(\mathbf{r}) = 0
\end{aligned}
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Proof that $Z_G(\mathbf{w}) \geq 0$

$\mathbf{w}^i = (w_1, w_2, \ldots, w_i, r_{i+1}, \ldots, r_n)$
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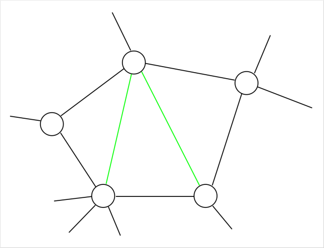
  - otherwise

$$\begin{aligned} Z_{G'}\left(\mathbf{w}^i\right) &= Z_{G' \setminus \{v_i\}}\left(\mathbf{w}^i\right) - w_{v_i} Z_{G' \setminus N+(v_i)}\left(\mathbf{w}^i\right) \\ &\geq Z_{G' \setminus \{v_i\}}\left(\mathbf{w}^{i-1}\right) - r_{v_i} Z_{G' \setminus N^+(v_i)}\left(\mathbf{w}^{i-1}\right) = Z_{G'}\left(\mathbf{w}^{i-1}\right) \geq 0 \end{aligned}$$

Perfect Strategy - conclusion

$Z_G(\mathbf{w}) \geq 0$ for every $0 \leq \mathbf{w} \leq \mathbf{r} \iff \mathbf{r}$ lies in the closure of $\mathcal{R}(G)$. Since $\mathbf{r}$ cannot lie inside $\mathcal{R}(G)$, it must belong to the boundary of the set $\mathcal{R}(G)$.

Chordal Graphs



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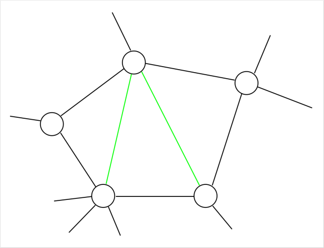

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Clique tree of $G = (V, E)$ is a tree $T = (V', E')$ such that

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- $\forall_{v \in V}$ the vertices of T containing v induces a connected subtree.

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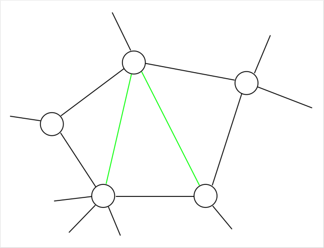

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G is chordal $\iff$ there exists a clique tree of G.

Chordal Graphs

Theorem. Let $G = (V, E)$ be a chordal graph and let $\mathbf{r} = (r_v)_{v \in V} \in ([0,1] \cap \mathbb{Q})^V$. If $\mathbf{r} \notin \mathcal{R}(G)$ then

$$\exists_{\mathbf{g},\mathbf{h} \in \mathbb{N}^V} \ \forall_{v \in V} \ \ g_v / h_v \leq r_v$$

and the game $(G, \mathbf{h}, \mathbf{g})$ is winning.

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$$\mathbf{r} \notin \mathcal{R}(G) \implies \exists_{0 \le \mathbf{w} \le \mathbf{r}} \; Z_G(\mathbf{w}) \le 0$$

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Case 1. G is a complete graph.

$Z_G(\mathbf{w}) \leq 0 \implies \sum_{v \in V} w_v \geq 1 \implies \sum_{v \in V} r_v \geq \sum_{v \in V} w_v \geq 1.$

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We take $\mathbf{g}, \mathbf{h} \in \mathbb{N}^V$ such that $g_v / h_v = r_v$ for each v. The game $(G, \mathbf{h}, \mathbf{g})$ is winning, since $\sum g_v / h_v \geq 1$.

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Idea. Find winning games for $G[V \setminus R]$ and $G[C]$ and combine them into final game.

Proof - winning game for $G[V \setminus R]$

If $\sum_{v \in C} r_v \geq 1$, then the game is winning on $G[C]$.

Assume that $\sum_{v \in C} r_v < 1 \implies \sum_{v \in C} w_v < 1$

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We define vectors $\mathbf{w}' = (w'_v)_{v \in V \setminus R}$ and $\mathbf{r}' = (r'_v)_{v \in V \setminus R}$ as

$$w'_v = \begin{cases} w_v/\alpha_w & \text{if } v \in S, \\ w_v & \text{otherwise, and} \end{cases} \qquad r'_v = \begin{cases} r_v/\alpha_r & \text{if } v \in S, \\ r_v & \text{otherwise,} \end{cases}$$

where $\alpha_r = 1 - \sum_{v \in R} r_v$ and $\alpha_w = 1 - \sum_{v \in R} w_v$

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where $\alpha_r = 1 - \sum_{v \in R} r_v$ and $\alpha_w = 1 - \sum_{v \in R} w_v$

It turns out that $\mathbf{w}' \leq \mathbf{r}'$ are vectors of numbers from $[0, 1]$.

Furthermore,

$$Z_{G'}\left(\mathbf{w}'\right) = Z_G(\mathbf{w})/\alpha_w \implies Z_{G'}\left(\mathbf{w}'\right) \leq 0 \implies \mathbf{r}' \notin \mathcal{R}\left(G'\right)$$

Thus, we can apply induction to find winning game $(G', \mathbf{h}', \mathbf{g}')$.

Proof - winning game for $G[C]$

Let G'' be the clique $G[C]$ with S contracted to a single vertex u.

We define the vector $\mathbf{r}'' = (r''_v)_{v \in R \cup \{u\}}$ as

$$r''_v = \begin{cases} r_v & \text{if } v \in R, \\ \alpha_r & \text{if } v = u \end{cases}$$

$r''_u + \sum_{v \in R} r''_v = 1 \implies \exists\, \mathbf{h}'', \mathbf{g}'' \in \mathbb{N}^V$ such that $g''_v / h''_v = r_v$ for every v and the game $(G'', \mathbf{h}'', \mathbf{g}'')$ is winning.

G is precisely the clique join of G' and G'' with respect to S and u.

Chordal graphs - conclusion

$U_G(x) :=$ polynomial obtained by plugging x for each variable x_v of Z_G.

Corollary. For any chordal graph G, the fractional hat chromatic number $\hat{\mu}(G)$ is equal to $1/r$ where r is the smallest positive root of $U_G(x)$.