Lower Bounds on the On-line Chain Partitioning of Semi-orders with Representation

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Thursday 19$^{\text{th}}$ May, 2022

Chain Partitioning

Dilworth's theorem

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On-line chain partitioning

An on-line chain partitioning algorithm is presented with a poset (X, P) in set order of elements $x_1, x_2, \ldots, x_n$ and constructs an on-line chain partitioning.

On line width

Definition

On-line width $olw(\omega)$ of the class of posets with width $\leq \omega$ is the largest $k \in \mathbb{N}$, such that there exists a strategy that forces any on-line chain partitioning algorithm to use k chains.

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The exact value of $olw(\omega)$ is unknown for $\omega > 2$.

Bounds on on-line width

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Bosek, Felsner, Kloch, Krawczyk, Matecki and Micek 2012

Improved lower bound of $(2 - o(1))\binom{\omega+1}{2}$ chains.

Interval Order

Interval order

We call a poset (X, P) an interval order if there exists a funciont I which maps each element $x \in X$ to a closed real number interval $I(x) = [l_x, r_x]$, such that for every $x_1, x_2 \in X$ it holds that $x_1 < x_2$ iff $r_{x_1} < l_{x_2}$.
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Semi-order

An interval order (X, P) is a semi-order if there is an interval representation I of (X, P) with unit-length intervals $[r_x - 1, r_x]$ on the real line. For each $x_1, x_2 \in X$, $x_1 < x_2$ if and only if $r_{x_1} < r_{x_2} - 1$.

On-Line Width of Interval Orders

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Kierstead and Trotter showed that $olwi(\omega) = 3\omega - 2$.

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On-Line Width of Interval Orders with Representation

On-line width $olwi_R(\omega)$ of the class of interval orders with width $\leq \omega$ is the largest $k \in \mathbb{N}$, such that there exists a strategy that forces any algorithm to use k chains to partition an interval order of witdth ω presented as intervals.
Instead of presenting the poset as points, it's presented as intervals, which provide an interval representation of a specific poset (X, P). Showed by Chrobak and Ślusarek to be $olwi_R(\omega) = 3\omega - 2$.

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On-line width $olws(\omega)$ of the class of semi-orders with width $\leq \omega$ is the largest $k \in \mathbb{N}$, such that there exists a strategy that forces any chain partitioning algorithm to use k chains.
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On-Line Width of Semi-Orders with Representation

On-line width $olws_R(\omega)$ of the class of interval orders with width $\leq \omega$ is the largest $k \in \mathbb{N}$, such that there exists a strategy that forces any algorithm to use k chains to partition when presented with a poset represented with unit-length intervals.
Exact value of $olws_R(\omega)$ is unknown.

On-Line Width of Semi-Orders with Representation

The problem first considered by Chrobak and Ślusarek. As previously mentioned, they showed that first-fit uses at most $2\omega - 1$ chains to perform an on-line chain partitioning. They also showed that any greedy algorithm can be forced to use $2\omega - 1$ chains.

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In 2005, Epstein and Levy showed a strategy, which for any positive integer k forces on-line algorithms to use $3k$ chains to partition a semi-order of width $2k$ represented with intervals.
This given us the best bounds known so far:

$$\lfloor \frac{3}{2}\omega \rfloor \leq olws_R(\omega) \leq 2\omega - 1$$

Theorem

We will show a slightly improved lower bound for on-line width of semi-orders with representation.

$$olws_R(\omega) \geq \lceil \frac{3}{2}\omega \rceil$$

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For a given k we will force any on-line chain partitioning algorithm to use $3k + 2$ chains for a poset of width $\omega = 2k + 1$.

Stage 1

We start with k identical intervals $x_1, \ldots, x_k$ with $x_i = 0$ for each $i \in \{1, \ldots, k\}$.

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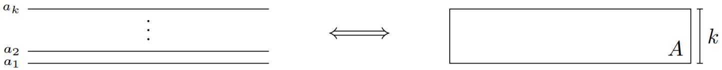

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Let B be the set of new chains used by the algorithm. We continue steps 2-5 until $|B| = k + 1$.

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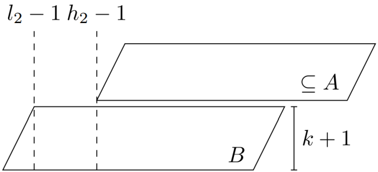

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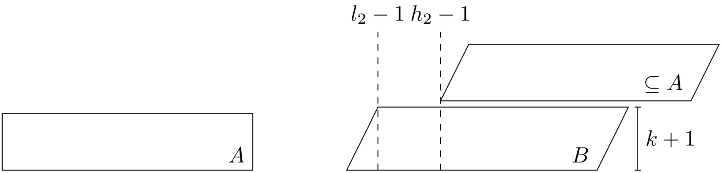

Because $1 < x_i < 2$, all the intervals created in this stage form an antichain of size at most ω.
All of them have to be assigned to different chains. At most k are in A and exactly $k + 1$ in B.

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For all new intervals x_i in this stage we have $-2 < x_i < -1$, so they form an antichain of size at most ω.

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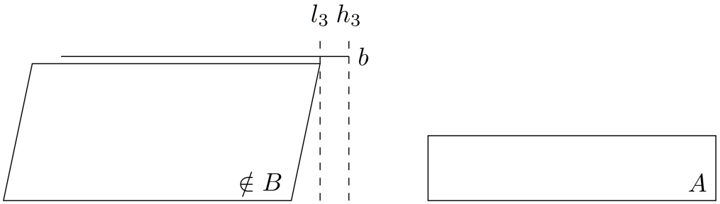
If the intervals are assigned to $k+1$ new chains, then we can finish.

We can assume that we finish the stage with an interval $x_B = h_3$ assigned to a chain $b \in B$



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The new intervals form an antichain of size at most $k + 1$. They all had to be assigned to different chains.
None of the those chains are in $A \cup \{b\}$.
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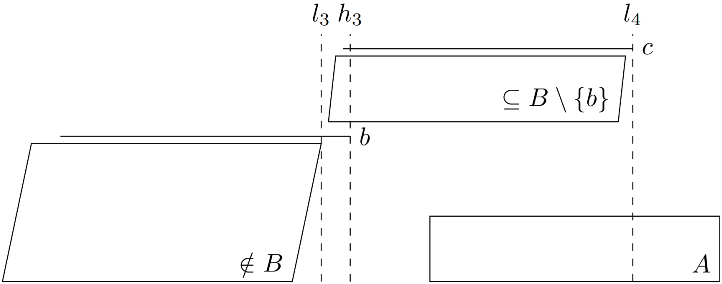

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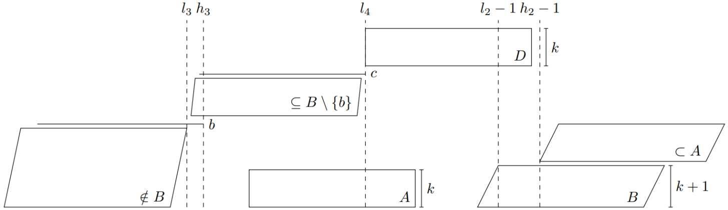
We want to show that the width ω has not been exceeded.



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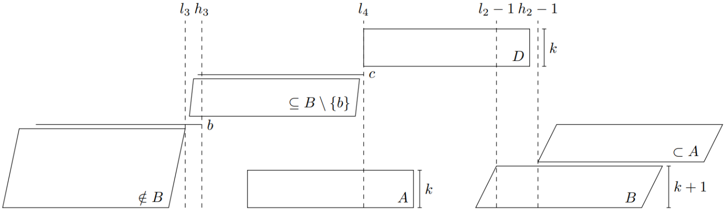

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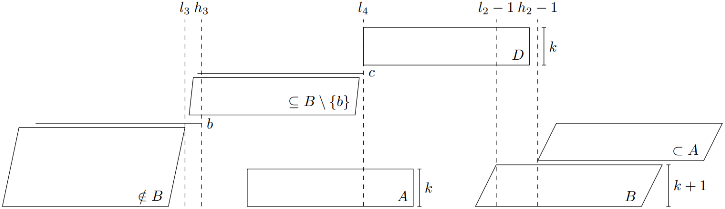

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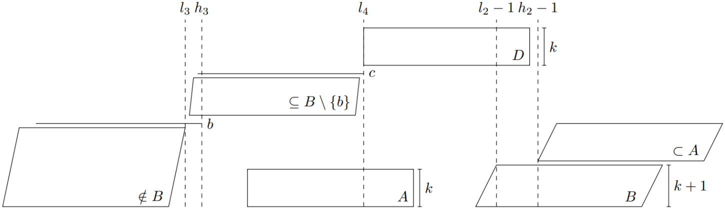

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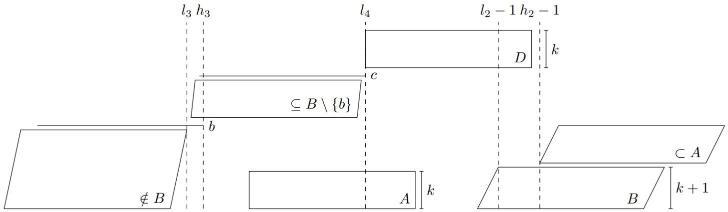

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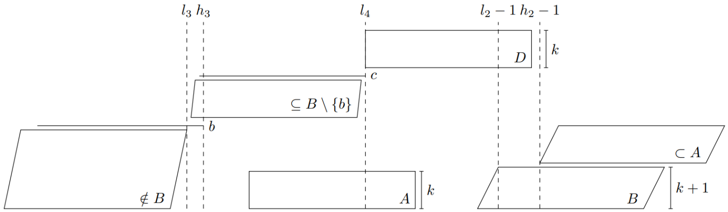

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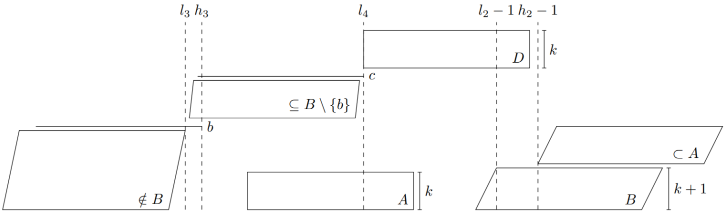
$$x_i = x_C + 1$$

From which we can deduce $l_2 - 1 < x_i < h_2 - 1$. Meaning the intervals from stage 2 incomparable to x_i are exactly the $k + 1$ intervals which were assigned to chains from B.

Stage 5



Let D denote the set of new chains used by the algorithm in stage 5.

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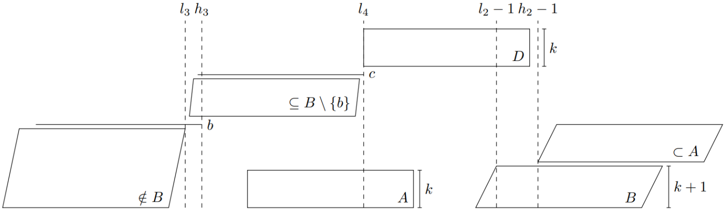

Let D denote the set of new chains used by the algorithm in stage 5. The total number of chains forced by our strategy is thus:

$$|A| + |B| + |\{c\}| + |D| = k + (k+1) + 1 + k = 3k + 2$$

Recap

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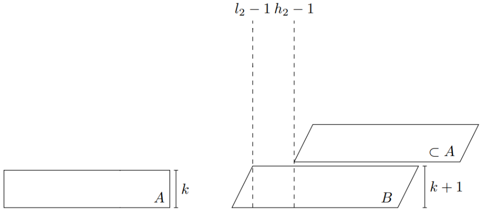

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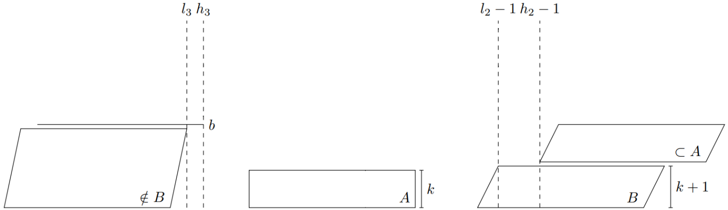

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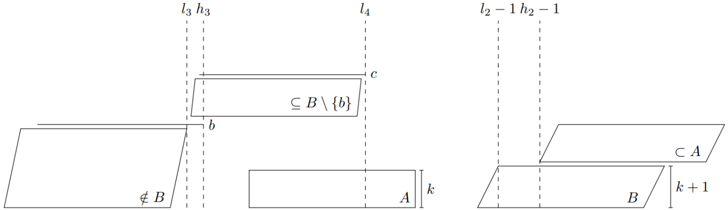

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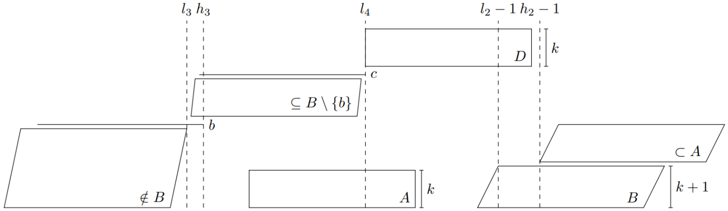
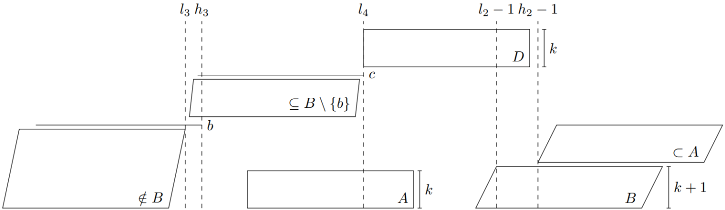
4. We move the window by 1 and add intervals until we get a new chain c.

Recap



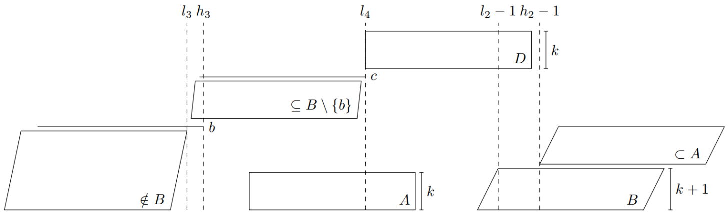
1. We force k chains A with intervals ending at 0.
2. We add new intervals in the $(1, 2)$ window until we force $k + 1$ new chains - the set B.
3. We move the window to the left by 3 and add intervals until one of them is assigned to a chain in B.
4. We move the window by 1 and add intervals until we get a new chain c.
5. We add k new intervals 1 to right of x_C. Each of them forces a new chain, we denote them D.

Conclusion



In the end we forced $3k + 2$ chain in $A \cup B \cup \{c\} \cup D$ while keeping the width of the poset to at most $2k + 1$.

Conclusion



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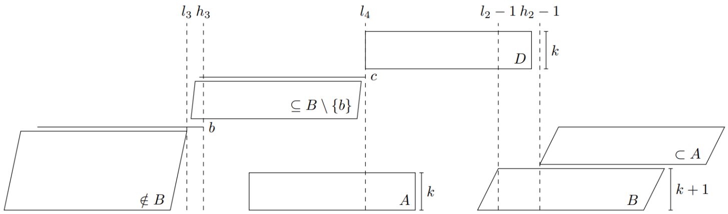
That means we have a new best lower bound for the on-line width of semi-orders with representation:

$$\lceil \frac{3}{2}\omega \rceil \leq olws_R(\omega)$$

Conclusion



In the end we forced $3k+2$ chain in $A \cup B \cup \{c\} \cup D$ while keeping the width of the poset to at most $2k+1$.
That means we have a new best lower bound for the on-line width of semi-orders with representation:

$$\lceil \frac{3}{2}\omega \rceil \leq olws_R(\omega) \leq 2\omega - 1$$

By including the upper bound we can give the exact value for $\omega = 3$:

$$olws_R(3) = 5$$

References

Contents and illustrations taken from

[1] C. Biró and I. R. Curbelo, *Improved lower bound on the on-line chain partitioning of semi-orders with representation*, 2021. DOI: 10.48550/ARXIV.2111.04790. [Online]. Available: https://arxiv.org/abs/2111.04790.