A better lower bound on average degree of 4-list-critical graphs[1]

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[1] Landon Rabern. *A better lower bound on average degree of 4-list-critical graphs.* 2016. DOI: 10.48550/ARXIV.1602.08532. URL: https://arxiv.org/abs/1602.08532.

Introduction

def. ***k-list-critical graph***

We call graph G *k-list-critical* if G is not $(k-1)$-choosable (doesn't always have a proper k list coloring), but every proper subgraph of G is $(k-1)$-choosable.

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Theorem (Main Theorem)

Every incomplete k-list-critical graph has average degree at least

$$k - 1 + \frac{k-3}{k^2 - 2k + 2}$$

Previous best bounds

	k-critical				k-list-critical			
k	Gallai	Kriv	KS	KY	KS	KR	CR[2]	Here
4	3.0769	3.1429	–	3.3333	–	–	–	**3.1000**
5	4.0909	4.1429	–	4.5000	–	4.0984	4.1000	**4.1176**
6	5.0909	5.1304	5.0976	5.6000	–	5.1053	5.1076	**5.1153**
7	6.0870	6.1176	6.0990	6.6667	–	6.1149	**6.1192**	6.1081
8	7.0820	7.1064	7.0980	7.7143	–	7.1128	**7.1167**	7.1000
9	8.0769	8.0968	8.0959	8.7500	8.0838	8.1094	**8.1130**	8.0923
10	9.0722	9.0886	9.0932	9.7778	9.0793	9.1055	**9.1088**	9.0853

Table 1: Previous known lower bounds on the average degree of k-critical and k-list-critical graphs

[2] Daniel Cranston and Landon Rabern. "Edge Lower Bounds for List Critical Graphs, Via Discharging". In: *Combinatorica* 38 (Feb. 2016). DOI: 10.1007/s00493-016-3584-6.

Notation

For graph $G = (V, E)$ we follow notation from the article where:

1. $|G| := |V(G)|$
2. $||G|| := |E(G)|$
3. $||A, B|| := |\{uv \in E : u \in A \wedge v \in B\}|$,
   For $A, B \subseteq V(G), A \cap B = \varnothing$

Gallai tree

def. *Gallai tree*

A graph T is called Gallai tree if every block (2-connected component) is either a clique or an odd cycle.

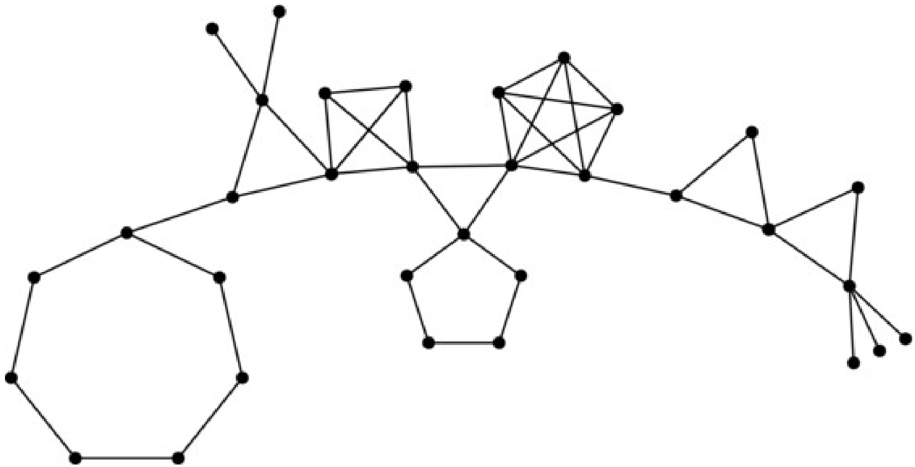

Figure 1: A Gallai tree with 15 blocks[3]

[3] Daniel Cranston and Landon Rabern. "Brooks' Theorem and Beyond". In: *Journal of Graph Theory* 80 (Dec. 2014). DOI: 10.1002/jgt.21847.

Lemma

Theorem (Gallai 1963)

In a k-critical graph, the vertices of degree $k-1$ induce a disjoint union of Gallai trees.

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For graph G, let $\beta_k(G)$ be equal to the independence number of the subgraph of G induced on the vertices of degree $k-1$.

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For graph G, let $\beta_k(G)$ be equal to the independence number of the subgraph of G induced on the vertices of degree $k-1$.

Lemma 1.

If $k \geq 4$ and $T \neq K_k$ is a Gallai tree with maximum degree at most $k-1$, then

$$2||T|| \leq (k-2)|T| + 2\beta_k(T)$$

Proof of Lemma 1.

Suppose the lemma is false. Choose a counterexample T with minimal $|T|$. Cases where T has 2 or less blocks are trivial. Suppose that T has at least 3 blocks.

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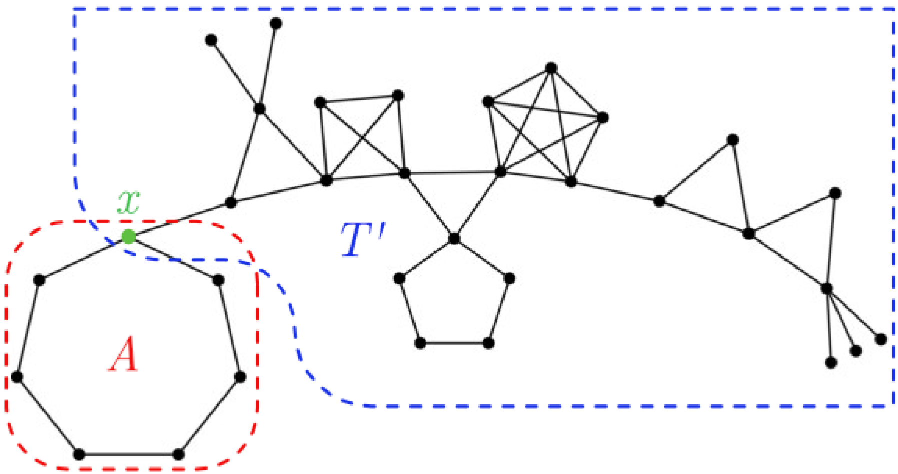

Figure 2: Example A, T'

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$$2||T'|| \leq (k-2)|T'| + 2\beta_K(T')$$

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$$2||T|| - 2||A|| \leq (k-2)(|T| + 1 - |A|) + 2\beta_K(T')$$

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Since T is a counter example:

$$\begin{aligned} -2||T|| + 2||A|| &\geq (k-2)(-|T| - 1 + |A|) - 2\beta_K(T') \\ +2||T|| &> (k-2)|T| + 2\beta_K(T) \\ \hline 2||A|| &> (k-2)(|A| - 1) + 2\beta_K(T) - 2\beta_K(T') \end{aligned}$$

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If $k > 4$ then $A = K_{k-1}$, if $k = 4$ then A is an odd cycle. In both cases:

$$d_T(x) = k - 1$$

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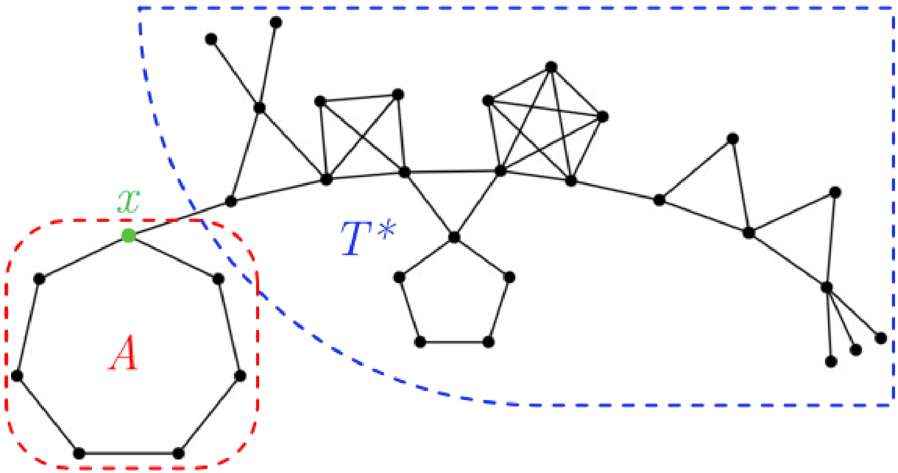

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We proved that $d_T(x) = k - 1$. But in T^* all of neighbours of x have degree at most $k-2$, so some vertex in $\{x\} \cup N(x)$ is in maximum independent set of degree $k-1$ vertices in T. Which implies that $\beta_k(T^*) \leq \beta_k(T) - 1$.

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This is a contradiction for $k \geq 4$. $\square$

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Hence:

$$\begin{aligned} 2||G|| &= 2||\mathcal{H}|| + 2||\mathcal{H}, \mathcal{L}|| + 2||\mathcal{L}|| \\ &= 2||\mathcal{H}|| + 2((k-1)|\mathcal{L}| - 2||\mathcal{L}||) + 2||\mathcal{L}|| \\ &= 2||\mathcal{H}|| + 2(k-1)|\mathcal{L}| - 2||\mathcal{L}|| \\ &\geq 2||\mathcal{H}|| + k|\mathcal{L}| - 2\beta_k(\mathcal{L}) \end{aligned}$$

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Hence: $\beta_k(\mathcal{L}) \geq ||\mathcal{H}|| + \frac{k}{2}|\mathcal{L}| - ||G||$

Proof of the Main Theorem - continued

def. *maximum independent cover number* (mic(G))

mic(G) $:= \max_I ||I, V(G) \setminus I||$, Where I is a independent set of G

We will use lemma proved by Kierstead[4].

Theorem (Kernel Magic)

Every k-list-critical graph G satisfies:

$$2||G|| \geq (k-2)|G| + \text{mic}(G) + 1$$

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$$\begin{aligned} 2||G|| &\geq (k-2)|G| + mic(G) + 1 \\ &\geq (k-2)|G| + M + (k-1)\beta_k(\mathcal{L}) + 1 \\ &\geq (k-2)|G| + M + (k-1)(||\mathcal{H}|| + \tfrac{k}{2}|\mathcal{L}| - ||G||) + 1 \end{aligned}$$

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Proof of the Main Theorem - continued

We now want to bound $M + (k-1)||\mathcal{H}||$. We denote:

- $\mathcal{C}$ is the set of all connected components of $G[\mathcal{H}]$
- $\alpha(C)$ is the independent number of C for any $C \in \mathcal{C}$
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We use two simple facts.

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$\min\{d_G(v) : v \in V(G)\} = k - 1$

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$$M \geq \sum_{C \in \mathcal{C}} k\alpha(C)$$

Because $\forall_{v \in M} d_G(v) \geq k$

$$\geq \sum_{C \in \mathcal{C}} k \frac{|C|}{\chi(C)}$$

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$$M + (k-1)||\mathcal{H}|| \geq \sum_{C \in \mathcal{C}} k \frac{|C|}{\chi(C)} + (k-1)||C||$$

We now will bound $k\frac{|C|}{\chi(C)} + (k-1)||C||$. If $\mathcal{L}$ is empty then by the fact 1 the average degree is at least k so the theorem holds true. Assume $\mathcal{L} \neq \varnothing$. Then for every $C \in \mathcal{C}$, $G[C]$ is $(k-1)$-colorable by k-list-criticality of G. So $\chi(C) \leq k-1$.

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- If $|C| = 1$ then the statement is true.
- If C is not a tree then $||C|| \geq |C|$, hence

$$k\frac{|C|}{\chi(C)} + (k-1)||C|| \geq k\frac{|C|}{k-1} + (k-1)|C| \geq (k-\frac{1}{2})|C|$$

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- If C is a tree then $\chi(C) \leq 2$, $||C|| = |C| - 1$, so

$$k\frac{|C|}{\chi(C)} + (k-1)||C|| \geq k\frac{|C|}{2} + (k-1)(|C|-1) \geq (k-\frac{1}{2})|C|$$

Lastly, we need a basic bound on $|\mathcal{L}|$. It is easy to see that.

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We put number k on every vertices in G and subtract from it the $d_G(v)$.

- The vertices in $\mathcal{L}$ end up with number 1 because $\forall_{v \in \mathcal{L}} d_G(v) = k - 1$
- The vertices in $V(G) \setminus \mathcal{L}$ end up with a non-positive number because $\forall_{v \in \mathcal{L}} d_G(v) \geq k$

Proof of the Main Theorem - continued

In summary we proved that for k-list-critical graph G the following inequalities are true

1. $(k+1)||G|| \geq (k-2)|G| + M + (k-1)||\mathcal{H}|| + \frac{k(k-1)}{2}|\mathcal{L}| + 1$
2. $M + (k-1)||\mathcal{H}|| \geq \sum_{C \in \mathcal{C}} k\frac{|C|}{\chi(C)} + (k-1)||C||$
3. $k\frac{|C|}{\chi(C)} + (k-1)||C|| \geq (k - \frac{1}{2})|C|$
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After combining all o them we get that

$$2||G|| \geq \left(k - 1 + \frac{k-3}{k^2-2k+2}\right)|G| + \frac{2}{k^2-2k+2}$$

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4. $|\mathcal{L}| \geq k|G| - 2||G||$

After combining all o them we get that

$$2||G|| \geq \left(k - 1 + \frac{k-3}{k^2-2k+2}\right)|G| + \frac{2}{k^2-2k+2}$$

This proves that

$$\frac{2||G||}{|G|} \geq k - 1\frac{k-3}{k^2-2k+2}$$

Which means the average degree of a k-list-critical graph is at least $k - 1 + \frac{k-3}{k^2-2k+2}$. $\square$

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