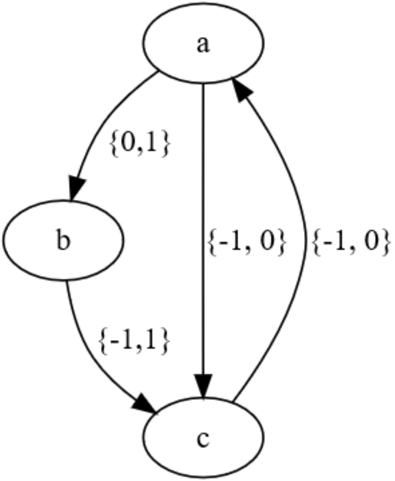
Digraphs are 2-weight choosable.

(based on work of Mahdad Khatirinejad, Reza Naserasr, Mike Newman, Ben Seamone, Brett Stevens)


a
{0,1}
b
{-1, 0}
{-1, 0}
{-1,1}
c

Th 1 (Combinatorial Nullstellensatz). Let F be an arbitrary field, and let $f = f(x_1, \ldots, x_n)$ be a polynomial in $F[x_1, \ldots, x_n]$. Suppose the total degree of f is $\sum_{i=1}^{n} t_i$, where each t_i is a nonnegative integer and suppose the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ in f is nonzero. If S_1, $\ldots S_n$ are subsets of F with $|S_i| > t_i$ then there eare $s_1 \in S_1$, $s_2 \in S_2$, $\ldots$, $s_n \in S_n$ so that

$$f(s_1, \ldots, s_n) \neq 0$$

Th 2 (Schur) If A is positive semi-definite Hermitian matrix, then $per(A) \geq det(A)$, with equality if and only if A is diagonal or A has a zero row.

$$det(M) = \sum_{s \in S_n} \left(sgn(s) \prod_{i=1}^{n} M_{i,s(i)} \right)$$

$$per(M) = \sum_{s \in S_n} \prod_{i=1}^{n} M_{i,s(i)}$$

M is positive semi-definite matrix iff $z^T M z \geq 0$ for real valued z.
Real valued M is Hermitian if $M = M^T$.

$$P_v = \sum_{e_h = v} x_e - \sum_{e_t = v} x_e$$

$$P_D = \prod_e (P_{e_h} - P_{e_t})$$

$$P_{e_h} - P_{e_t} = \sum_{e'} M_{e,e'} x_{e'}$$

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$$M_{e,e'} = 2 \text{ if } e = e'$$

$$M_{e,e'} = -2 \text{ if } e = o(e')$$

$$M_{e,e'} = 1 \text{ if } e \cap e' = 1$$

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$$M_{e,e'} = 0 \text{ if } e \cap e' = \emptyset, e \cap o(e') = \emptyset$$

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$$\text{coef of } \prod_e x_e = per(M)$$

goal

$$per(M) > 0$$

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$$M = M^T$$

$$M = XX^T$$

$$X_{e,v} = 1 \text{ if } e_h = v$$

$$X_{e,v} = -1 \text{ if } e_t = v$$

$$X_{e,v} = 0 \text{ if } e_h \neq v, e_t \neq v$$

$$z^T X X^T z \geq 0$$

$$det(XX^T) \geq 0$$

When M is diagonal?
When M has zero row?

$$per(M) = per(XX^T) > det(XX^T) \geq 0$$

Every graph G has an orientation D such that the weighted degrees of D give proper vertex colouring of G.

Every digraph D has a subgraph F such that the weighted degrees of F give proper vertex colouring of D.

Thank you!