Separating polynomial χ-boundedness from χ-boundedness

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Notation

- $[n] = \{1, \ldots, n\}$
- $\mathbb{P} = \{p_1, p_2, \ldots\}$ is the set of all primes
- $\chi(G)$ denotes the chromatic number of graph G
- $\omega(G)$ denotes the clique number of graph G

χ-boundedness

Definition

A class of graphs $\mathcal{C}$ is χ-bounded if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $\chi(G) \leq f(\omega(G))$ for every graph $G \in \mathcal{C}$. A χ-bounded class $\mathcal{C}$ is polynomially χ-bounded if such a function f can be chosen to be a polynomial. A class $\mathcal{C}$ is hereditary if it is closed under taking induced subgraphs.

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A well-known and fundamental open problem, due to [Esperet, 2017], has been to decide whether every hereditary χ-bounded class of graphs is polynomially χ-bounded.

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A well-known and fundamental open problem, due to [Esperet, 2017], has been to decide whether every hereditary χ-bounded class of graphs is polynomially χ-bounded.

We provide a negative answer to this question. More generally, we prove that χ-boundedness may require arbitrarily fast growing functions.

Main result

Theorem

For every function $f : \mathbb{N} \to \mathbb{N}$*, there exists a hereditary* χ*-bounded graph class* $\mathcal{C}$ *which, for every* $n \geq 2$*, contains a graph* $G \in \mathcal{C}$ *such that* $\omega(G) \leq n$ *and* $\chi(G) \geq f(n)$*.*

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For every function $f : \mathbb{N} \to \mathbb{N}$, there exists a hereditary χ-bounded graph class $\mathcal{C}$ which, for every $n \geq 2$, contains a graph $G \in \mathcal{C}$ such that $\omega(G) \leq n$ and $\chi(G) \geq f(n)$.

The proof is heavily based on the idea used by [Carbonero, Hompe, Moore, Spirkl, 2022] in their recent solution to another well-known problem. They proved that for every $k \in \mathbb{N}$, there is a K_4-free graph G with $\chi(G) \geq k$ such that every triangle-free induced subgraph of G has chromatic number at most 4. Their proof, in turn, relies on an idea by [Kierstead, Trotter, 1992], who proved that the class of oriented graphs excluding an oriented path of length 3 as an induced subgraph is not χ-bounded.

Proof of Main Result

Lemma (2)

For every $k \in \mathbb{N}$, there is a triangle-free graph G_k and an acyclic orientation of its edges such that $\chi(G_k) = k$ and for every pair of vertices u and v, there is at most one directed path from u to v in G_k.

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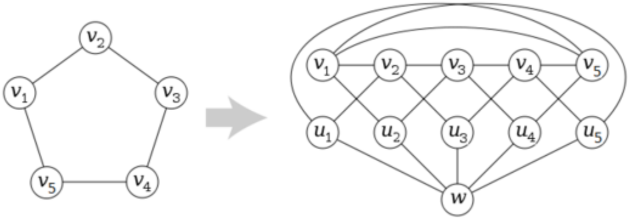
We can use any standard construction of triangle-free graphs with arbitrarily high chromatic number, for example Mycielskian (1955), and orient the edges in a way that follows naturally from the construction.



Source: Wikipedia

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Fix a function $f : \mathbb{N} \to \mathbb{N}$. Define a new function $g : \mathbb{P} \to \mathbb{N}$, given by

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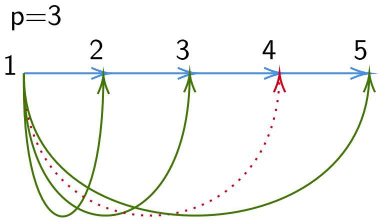

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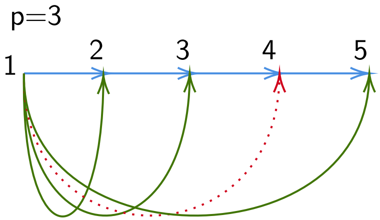

- $V(G'_p) := V(G_{g(p)})$;
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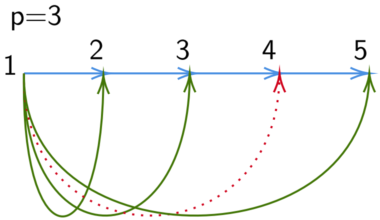

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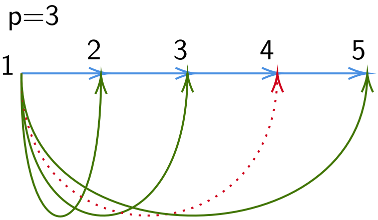

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- Therefore $\chi(G'_p) \geq \chi(G_{g(p)}) = g(p)$

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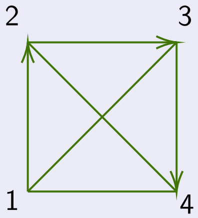

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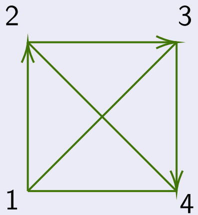

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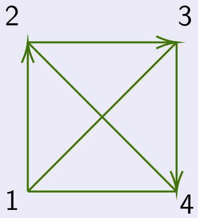

- Suppose C is a clique in G'_p of size $k > p$
- Let $v_1, \ldots, v_k$ be the vertices of C ordered so that $v_1 < \cdots < v_k$
- By Pigeonhole principle, there are some $i < j$ such that $d(v_1, v_i) \equiv_p d(v_1, v_j)$

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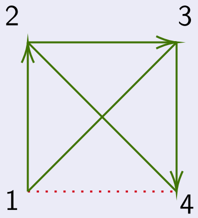

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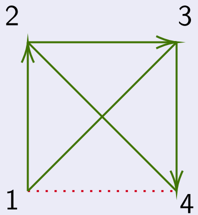

- There are $i < j$ such that $d(v_1, v_i) \equiv_p d(v_1, v_j)$
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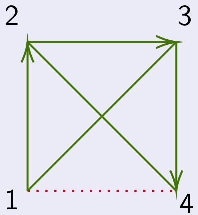

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- We conclude that $d(v_i, v_j) \equiv_p 0$, so $v_i v_j$ could not have been an edge of G'_p

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- To construct the class $\mathcal{C}$ that witnesses Theorem, we take the graphs G'_p for all primes p together with all their induced subgraphs
- The second part of the statement of Theorem follows: for every number $n \geq 2$, where $p = p_i \leq n < p_{i+1}$, the graph $G'_p \in \mathcal{C}$ satisfies $\chi(G'_p) \geq g(p) \geq f(n)$ and $\omega(G'_p) \leq p \leq n$

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- It remains to prove that the class $\mathcal{C}$ is χ-bounded

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Let p be a prime and G be an induced subgraph of G'_p with $n = \omega(G) < p$. Then $\chi(G) \le n^{n^2}$.

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This lemma indeed implies χ-boundedness. Consider the function $f' : \mathbb{N} \to \mathbb{N}$ defined as

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- $n = \omega(G) = p$, then $\chi(G) \leq \chi(G'_p) \leq f'(n)$, since $p \leq n$.

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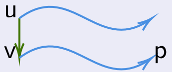

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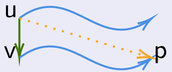
- Then u is a starting vertex of a directed path $u \to v \to \cdots \to p$ of length $c+1$, where p witnesses $c(v)$. Contradiction



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- We let $\Phi(n) = |F_n \setminus \{0\}|$. It is clear from the definition that

$$\Phi(n) \leq 1 + 2 + \cdots + n \leq n^2$$

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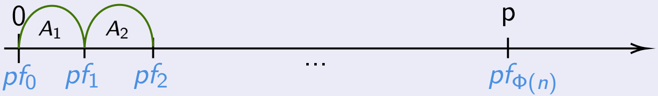

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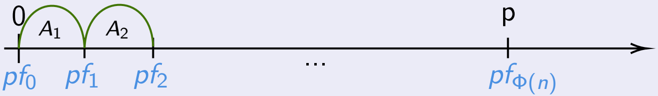

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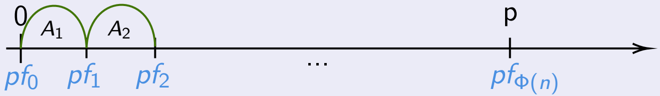
- Hence $\{pf_0, pf_1, \ldots, pf_{\Phi(n)}\} \cap [p-1] = \emptyset$, so $A_1, \ldots, A_{\Phi(n)}$ is a partition of $[p-1]$



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- Consequently, the sum of any m numbers in A_i lies in $(p(s-1), ps)$, so it never equals 0 modulo p, as required

□

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- It follows that $E_1, \ldots, E_{\Phi(n)}$ is a partition of the edge set of G'_p
- For each $i \in [\Phi(n)]$, let G_i^* be the subgraph of G obtained by restricting the edge set to E_i, keeping the orientations of these edges

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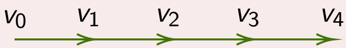

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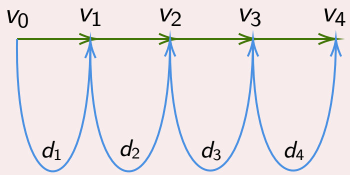

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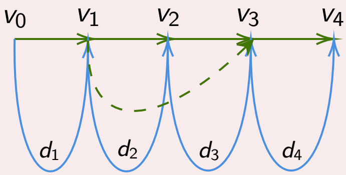
- For any $0 \leq i < j \leq n$, we have $d(v_i, v_j) = d(v_i, v_{i+1}) + \cdots + d(v_{j-1}, v_j)$



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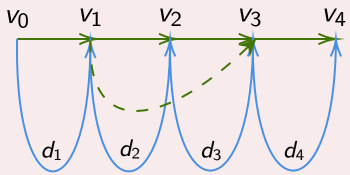
- Lemma 5 implies $d(v_i, v_j) \not\equiv_p 0$ (sum of m numbers in A_i), so $v_i v_j$ is an edge of G'_p, and thus of G



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- Thus $d(v_{i-1}, v_i) \in_p A_i$ for $1 \le i \le n$
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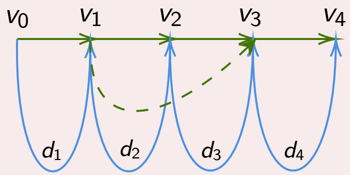
- Lemma 5 implies $d(v_i, v_j) \not\equiv_p 0$ (sum of m numbers in A_i), so $v_i v_j$ is an edge of G'_p, and thus of G
- We conclude that $\{v_0, v_1, \dots, v_n\}$ is a clique in G, which contradicts $\omega(G) = n$



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- Then the product colouring $c(v) := (c_1(v), \ldots, c_{\Phi(n)}(v))$ is $n^{\Phi(n)}$-colouring of G
- Since $\Phi(n) \leq n^2$, we conclude that $\chi(G) \leq n^{n^2}$

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The End