Brooks' Theorem via the Alon-Tarsi Theorem

Hladký, Král, Schauz

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Theoretical Computer Science

Brooks' Theorem

Theorem [Brooks 1941]

Let G be a connected graph. If G is neither a **complete graph** nor an **odd cycle**, then G is $\Delta(G)$-colorable.

List coloring

Δ-choosable

Graph G is called Δ-choosable if it can be colored from lists of length $\Delta(G)$.

degree-choosable

Graph G is called degree-choosable if it can be colored from lists,
such that list of vertex v has length $\deg(v)$.

Paintability

Consider the game on graph G with two players Marker and Remover. Initially each vertex is equipped with k erasers.

In each turn:

1. Marker marks non-empty subset of vertices.
2. Remover removes an independent subset of marked vertices from the graph and clears marks of other vertices with their erasers.

Remover wins if all the vertices are eventually removed from the graph. If Remover has a winning strategy, the graph is said to be $(k+1)$-paintable.

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Observation

If graph G is k-paintable, then G is also k-choosable.

Brooks' Theorem

Theorem [Brooks 1941]

Let G be a connected graph. If G is neither a **complete graph** nor an **odd cycle**, then G is $\Delta(G)$-colorable.

Theorem [Hladký, Král, Schauz 2010]

Let G be a connected graph. If G is neither a **complete graph** nor an **odd cycle**, then G is $\Delta(G)$-choosable and $\Delta(G)$-paintable.

Alon-Tarsi Theorem

For an oriented graph G we define:

- spanning subgraph of G – subgraph of G containing all vertices of G
- Eulerian subgraph of G – spanning subgraph of G,
  such that for each vertex v we have $\deg_{in}(v) = \deg_{out}(v)$.
- even Eulerian subgraph of G – Eulerian subgraph of G with **even** number of edges
- odd Eulerian subgraph of G – Eulerian subgraph of G with **odd** number of edges

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Theorem [Alon, Tarsi 1992]

Let G be an oriented graph. If the number of **even** and **odd** Eulerian subgraphs **differ**, then G can be colored from any lists $L(v)$, such that $|L(v)| \geq \deg_{in}(v) + 1$.

Alon-Tarsi Theorem

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Let G be an oriented graph. If the number of **even** and **odd** Eulerian subgraphs **differ**, then G can be colored from any lists $L(v)$, such that $|L(v)| \geq \deg_{in}(v) + 1$.

Theorem [Schauz 2010]

Let G be an oriented graph. If the number of **even** and **odd** Eulerian subgraphs **differ**, then Remover has a winning strategy if every vertex v is initially equipped with $\deg_{in}(v)$ erasers.

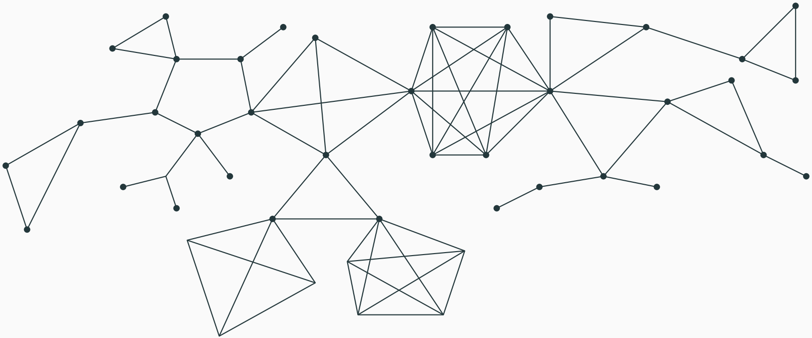
Main theorem

Theorem [Hladký, Král, Schauz 2010]

Let G be a connected graph. If G **is not a Gallai tree**,
then G is degree-choosable and degree-paintable.

Gallai tree

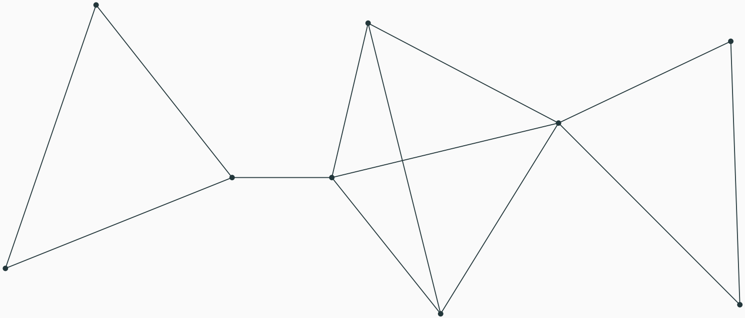
A **Gallai tree** is a graph in which every block is a complete graph or an odd cycle.



Gallai tree

Theorem

No Gallai tree is degree-choosable



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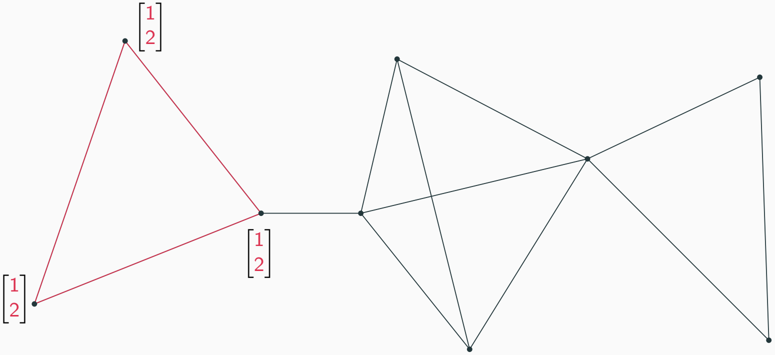

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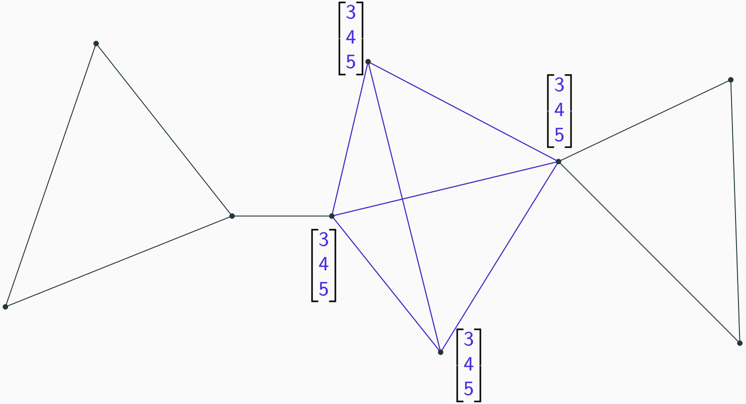

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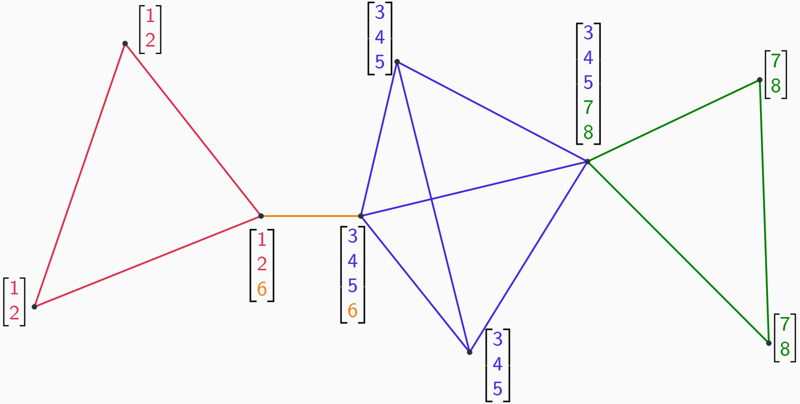

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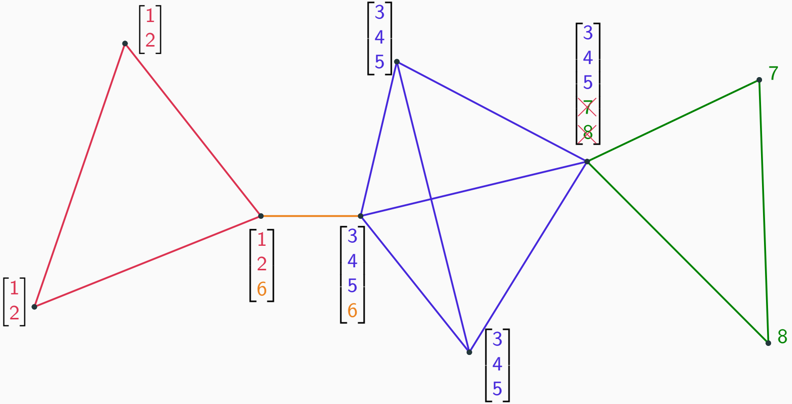


Structural lemma

Lemma

Every connected graph G that **is not a Gallai tree** has an induced subgraph that is an even cycle with at most one chord.

H – block that is neither a complete graph nor an odd cycle.

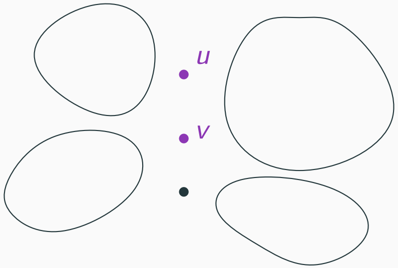
Let S be minimal vertex cut of block H. Then:

- S is proper
- $|S| \geq 2$

Structural lemma

S – minimal vertex cut of block H

u, v – any vertices in S

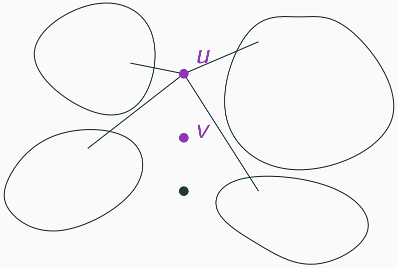


Components created after removing S

Structural lemma

S – minimal vertex cut of block H

u, v – any vertices in S



u, v are connected to every component of $G \setminus S$

Structural lemma

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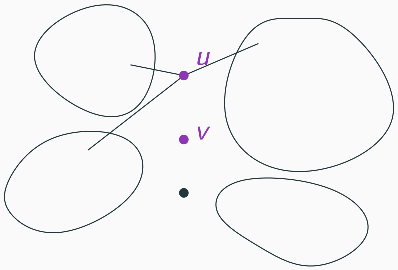

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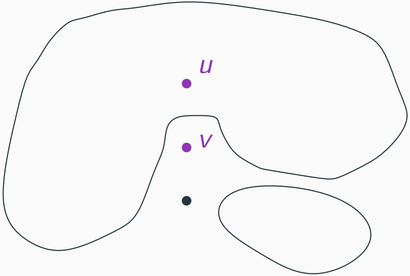

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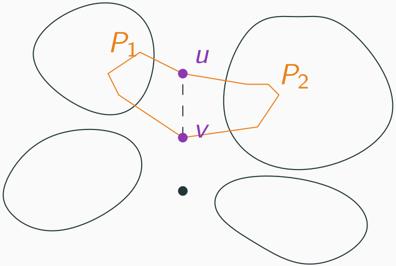
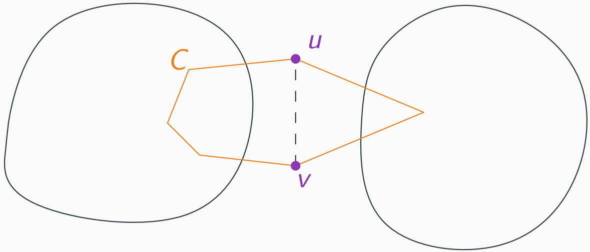


P_1, P_2 – shortest paths joining u, v through different components of $G \setminus S$

Paths P_1, P_2 create a cycle C with $|C| \geq 4$ and at most one chord uv.

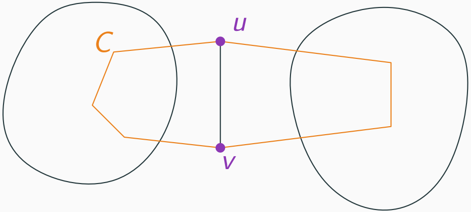
Structural lemma

C is even



Structural lemma

C is odd and has chord uv



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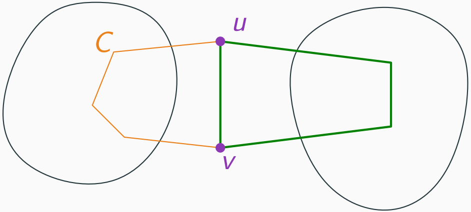
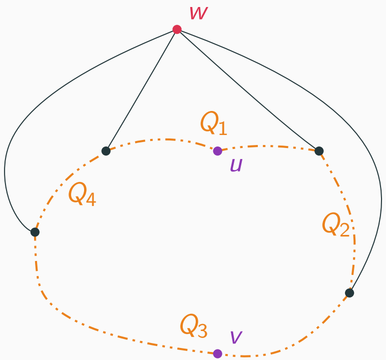


Structural lemma

C is odd, has no chords and there is vertex $w \notin C$, such that w has at least 2 neighbours on a cycle



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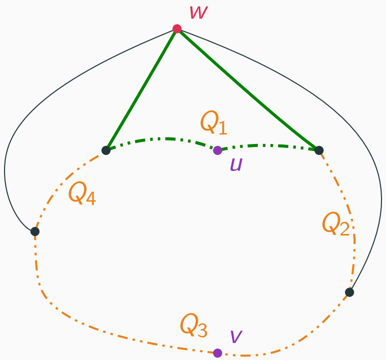

One of the paths Q_i has even length

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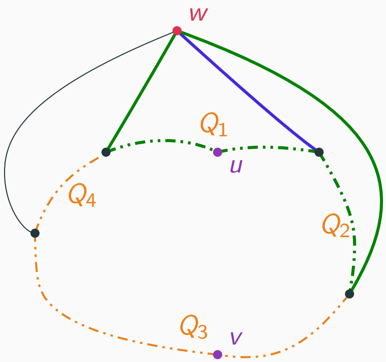

All paths Q_i have odd length (in that case w has at least 3 neighbours on cycle)

Structural lemma

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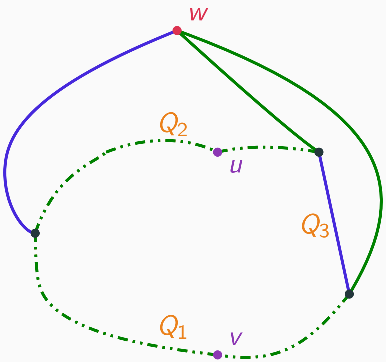
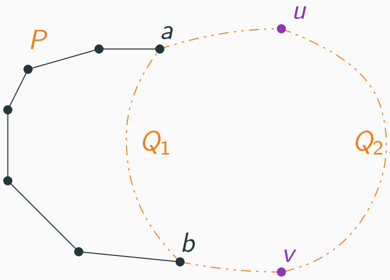


All paths Q_i have odd length (in that case w has at least 3 neighbours on cycle)

Structural lemma

C is odd, has no chords and every vertex $w \notin C$ has at most 1 neighbour on a cycle



P – shortest path connecting two vertices of C that is disjoint from C

Structural lemma

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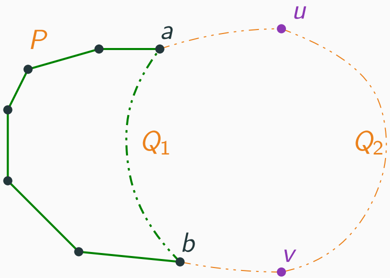

One of the $P \cup Q_1$, $P \cup Q_2$ is even cycle

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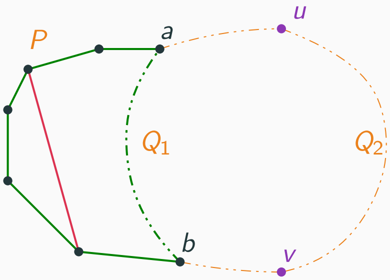

This even cycle is chordless.

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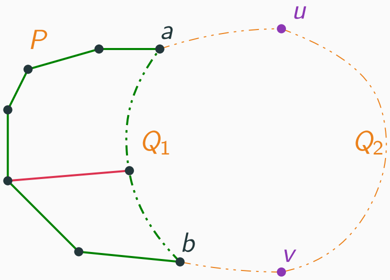

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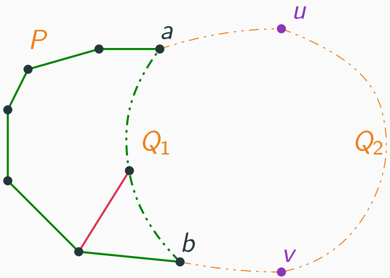

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Every connected graph G that **is not a Gallai tree** has an induced subgraph that is an even cycle with at most one chord.

Lemma

Let G be a connected graph and v an arbitrary vertex of G. The vertices of G can be ordered in such a way that every vertex except v is preceded by at least one of its neighbors.

Main theorem

Theorem [Hladký, Král, Schauz 2010]

Let G be a connected graph. If G **is not a Gallai tree**, then G is degree-choosable and degree-paintable.

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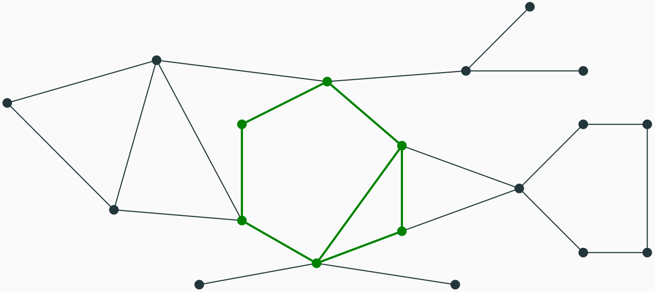
Let's recall:

Theorem [Schauz 2010]

Let G be an oriented graph. If the number of **even** and **odd** Eulerian subgraphs **differ**, then Remover has a winning strategy if every vertex v is initially equipped with $\deg_{in}(v)$ erasers.

Main theorem

By Structural Lemma, G contains an even cycle C with at most one chord.



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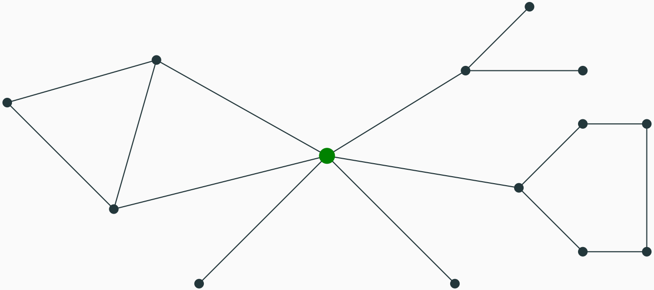

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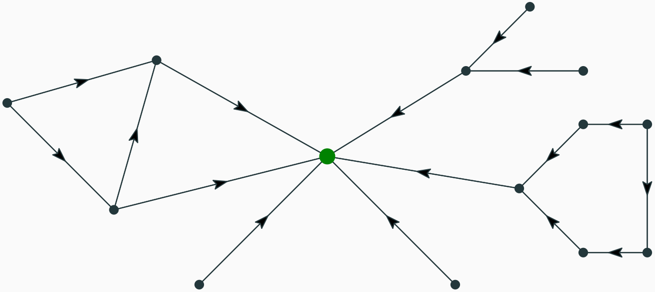
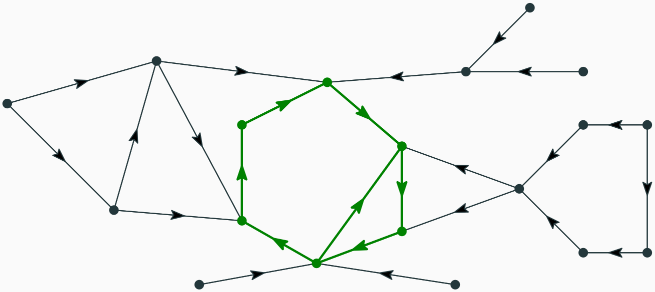


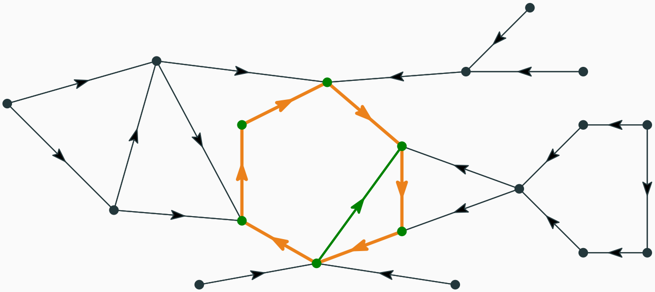
Main theorem

Every vertex of G has at least one outgoing edge.



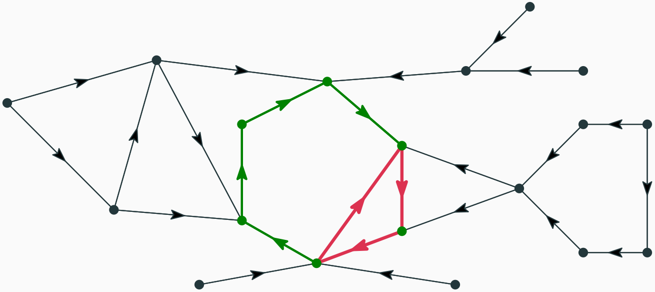
Main theorem

G contains at least 2 even Eulerian subgraphs – empty one and cycle C



Main theorem

G contains at most 1 odd Eulerian subgraph



Main theorem

From:

Theorem [Schauz 2010]

Let G be an oriented graph. If the number of **even** and **odd** Eulerian subgraphs **differ**, then Remover has a winning strategy if every vertex v is initially equipped with $\deg_{in}(v)$ erasers.

We get:

Theorem [Hladký, Král, Schauz 2010]

Let G be a connected graph. If G **is not a Gallai tree**, then G is degree-choosable and degree-paintable.

Brooks' Theorem – minimal counter-example

Theorem [Brooks 1941]

Let G be a connected graph. If G is neither a **complete graph** nor an **odd cycle**, then G is $\Delta(G)$-colorable.

Let G be minimal counter-example and let v be any vertex of G.

- $G \setminus v$ can be colored
- v must have $\deg(v) = \Delta$
- G is Δ-regular

Brooks' Theorem

Theorem [Hladký, Král, Schauz 2010]

Let G be a connected graph. If G is neither a **complete graph** nor an **odd cycle**, then G is $\Delta(G)$-choosable and $\Delta(G)$-paintable.

Let G be minimal counter-example. Then G is:

- neither a **complete graph** nor an **odd cycle**
- not Δ-paintable(choosable)
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Brooks' Theorem

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Let G be minimal counter-example. Then G is:

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- not Δ-paintable(choosable)
- Δ-regular

Then:

- if G is not a Gallai tree, then G is Δ-paintable(choosable)
- if G is a Gallai tree, then $G = K_{\Delta+1}$