Complete minors and average degree -- a short proof

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THEORETICAL COMPUTER SCIENCE
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Definition

An undirected graph H is called a minor of the graph G if H can be formed from G by:

- *deleting edges,*
- *deleting vertices,*
- *contracting edges.*

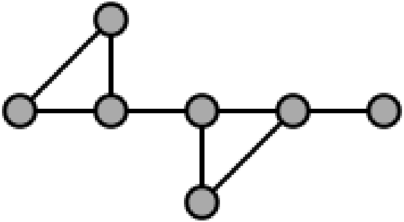


Figure: G

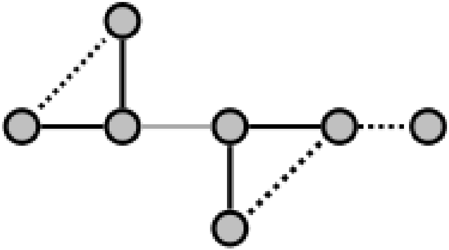


Figure: G with marked changes

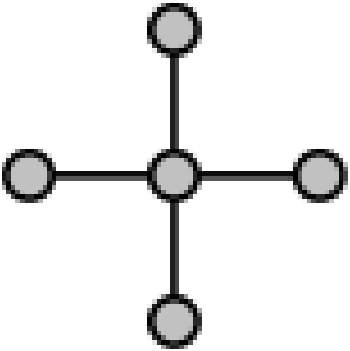


Figure: H

The Hadwiger conjecture in graph theory states that if G is loopless and has no K_t minor then its chromatic number satisfies $\chi(G) < t$. Currently we only know it is true for $1 \leq t \leq 6$.

In more detail, if all proper colorings of an undirected graph G use t or more colors, then one can find k disjoint connected subgraphs of G such that each subgraph is connected by an edge to each other subgraph. Contracting the edges within each of these subgraphs so that each subgraph collapses to a single vertex produces a complete graph K_t on t vertices as a minor of G.

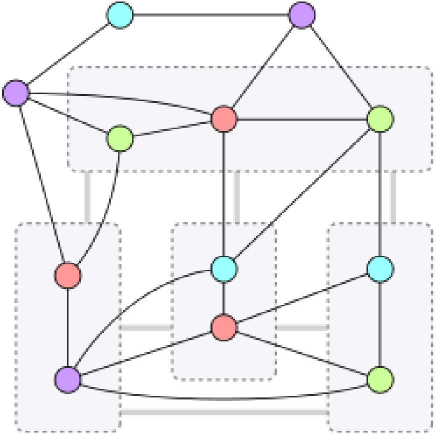
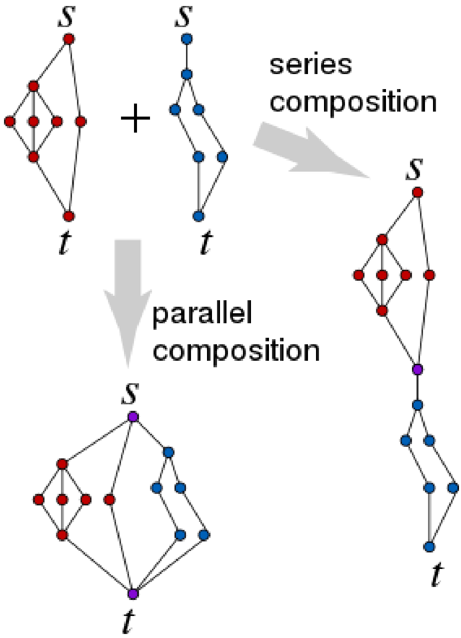


Figure: All proper colorings of above graph use at least 4 colors and it is possible to find K_4 as minor

Hadwiger proved its truth for $t = 4$. The K_4-minor-free graphs are the series–parallel graphs and their subgraphs.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Such graphs have a vertex with at most two incident edges, so they are $3-$colorable, by removing the such vertex, coloring the remaining graph, and coloring the vertex on a remaining color.

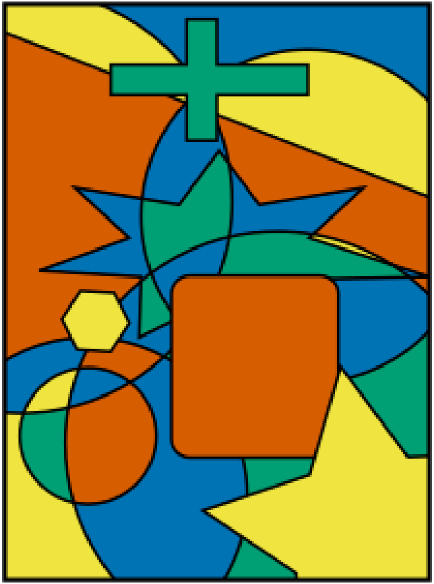

Figure: Four color theorem example

If the conjecture is true, then every graph which chromatic number is equal or greater than 5 contains K_5 minor and by Wagner's theorem is nonplanar. In 1937 Wagner showed that the case $t = 5$ is equivalent to the four color theorem, by proving that every graph which has no K_5 minor can be decomposed into components, which have chromatic number equal or less than 4, what shows the 4-colorability of a K_5-minor-free graph.

So we can observe that the conjecture is a generalization of the four-color theorem and for many it is one of the most important and challenging open problems in the graph theory.

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Kostochka and Thomason independently proved it to be $d/\sqrt{\log d}$. Tightness follows by looking into a random graph. Finally, Thomason found the asymptotic value of this extremal function.

The paper provides a short and self-contained proof of the Kostochka-Thomason bound.

Theorem

Let $G = (V, E)$ be a graph with $|E|/|V| \geq d$, where d is a sufficiently large integer. Then G contains a minor of the complete graph on at least $\frac{d}{10\sqrt{\ln d}}$ vertices.

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Lemma

Let $H = (V, E)$ *be a graph on at most* n *vertices with* $\delta(H) \geq n/6$*.*
Let $t \leq \frac{n}{\sqrt{\ln n}}$*, and let* $A_1, \dots, A_t \subset V$ *with* $|A_j| \leq \frac{n}{e^{\sqrt{\ln n}/3}}$*.*
Then there is $B \subset V$ *of size at most* $3.1\sqrt{\ln n}$ *such that* B *dominates all but at most* $\frac{n}{e^{\sqrt{\ln n}/3}}$ *vertices of* V*, and* $B \backslash A_j \neq \emptyset$ *for all* $j = 1, \dots, t$*.*

Proof.

Choose $s = 3.1\sqrt{\ln n}$ vertices of V independently at random with repetitions and call it B. For every $v \in V$

$$Pr[N(v) \cap B = \emptyset] \leq e^{-s/6}.$$

The expected number of vertices not dominated by B is by Markov's inequality $\leq ne^{-\sqrt{\ln n}/3}$ with probability $> 1/2$. Since $|V| > \delta(H) \geq n/6$, for every A_j

$$Pr[B \subseteq A_j] < \frac{1}{n}.$$

So $P[B \backslash A_j \neq \emptyset]$ for all j is $\geq 1 - 1/\sqrt{\ln n}$.
Union bound gives us requested result. □

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Theorem

Let $G = (V, E)$ *be a graph with* $|E|/|V| \geq d$*, where* d *is a sufficiently large integer. Then* G *contains a minor of the complete graph on at least* $\frac{d}{10\sqrt{\ln d}}$ *vertices.*

General proof plan:

- Let H be a subgraph of G which contains a $d/3$-connected subgraph H_0 with $\delta(H_0) \geq 2d/3$.
- Set $i = 0$ and repeat the following iteration $d/10\sqrt{\ln d}$ times. Let $H_i = (V_i, E_i) \subseteq H_0$ be the current graph.
- Let $A_1, \dots, A_{i-1}$ are subsets of V_i with $|A_j| \leq \frac{2d}{e^{\sqrt{\ln(2d)/3}}}$.
- H_i is connected, has $\delta(H_i) > d/3$ and the diameter of H_i is at most 14.
- Apply lemma with $H := H_i$, $n := 2d$, $t := i - 1$ and $A_1, \dots, A_{i-1}$ we get a subset B_i such that $|B_i| \leq 3.1\sqrt{\ln(2d)}$.
- Now turn B_i into a connected set by adding some more vertices of H_i.
- We obtain a connected subset B_i with $|B_i| \leq (3.1 + o(1))\sqrt{\ln(2d)}$, dominating all but at most $\frac{2d}{e^{\sqrt{\ln(2d)/3}}}$ vertices of V_i and connected to every previous B_j.

- Update $V_{i+1} := V_i - B_i$, $A_i := V_{i+1} - N_{H_i}(B_i)$, and $A_j := A_j \cap V_{i+1}$, $j = 1, \dots, i-1$, and proceed to the next iteration.
- Observe the total number of vertices deleted in all iterations satisfies:

$$\left| \cup_i B_i \right| < \frac{d}{3},$$

- Since we started with the $d/3$-connected graph H_0 with $\delta(H_0) \geq 2d/3$, each H_i is connected and has $\delta(H_i) > d/3$.
- After all iterations, we get a family of $d/10\sqrt{\ln d}$ branch sets B_i, all connected, and with an edge of H_0 between every pair of them. Hence they form a complete minor K_t with $t = d/10\sqrt{\ln d}$.

Bibliography:

- Noga Alon, Michael Krivelevich, Benny Sudakov, 2022, Complete minors and average degree -- a short proof, https://arxiv.org/abs/2202.08530
- wikipedia.org

All of the images where taken from wikipedia.org.