Note on a Lamp Lighting Problem

based on an article of H. Eriksson, K. Eriksson and J. Sjöstrand

Krzysztof Ziobro

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Introduction

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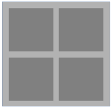

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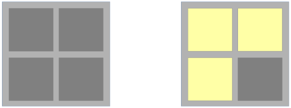

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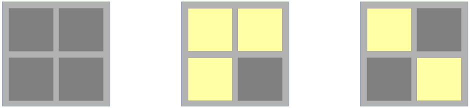

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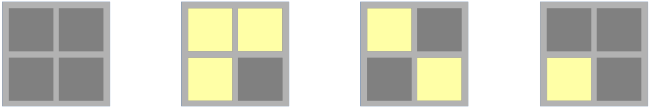

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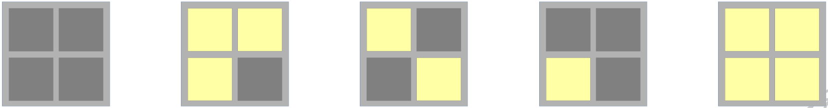

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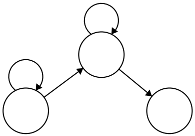

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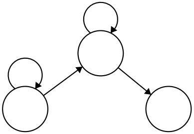

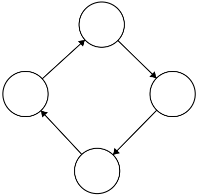

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They prove the following result:

Theorem

If G is a directed graph on vertex set V, such that for each odd subset $U \subseteq V$ there is a vertex with odd out-degree in the induced subgraph on U, then it is possible to light all lamps.

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Of course, we can light any subset of vertices of size n - we just "remove" one of the vertices and take advantage of our induction hypothesis.

However, we do not know whether the removed vertex v will end up lit up. If it does for any $v \in V(G)$, then we just managed to light up the whole graph.

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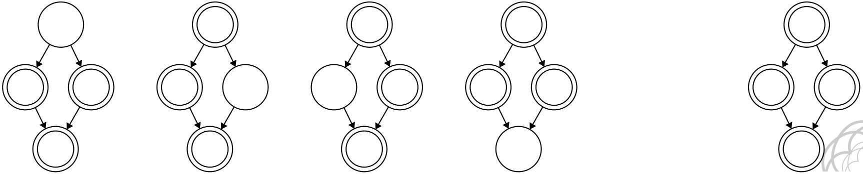
Then we have two cases:

- $n + 1$ is even: we can perform an action of lighting all vertices but v for all $v \in V(G)$. We see that every vertex has been lighted n times, and n is odd, so we end up with all vetrices lit.



Theorem for directed graphs: proof

- $n + 1$ is odd: then there is a vertex v which has an odd out-degree (every subgraph with odd number of vertices has one). Let's light $N(v)$. We know that $|V(G) - N(v)|$ is even, so we can, for every vertex of $u \in V(G) - N(v)$ perform an action of lighting all vertices of G but u (so we will perform $|V(G) - N(v)|$ of such actions).

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  Now we have lit every vertex of $N(v)$ odd $(|V(G)-N(v)|+1)$ number of times, and every vertex of $V(G)-N(v)$ odd $(|V(G)-N(v)|-1)$ number of times.

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Undirected case

Having proved the theorem we have an easy route to proving the result for case of undirected graphs with loops:

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Given any undirected graph G with loops on every vertex, it is always possible to light all of the vertices by a sequence of operations allowed in lamp lighting problem.

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Every bidirected edge adds two directed edges, so we end up with odd number of edges in every odd induced subgraph - that means at least one vertex in this subgraph has to have an odd out-degree.

Lighting as many lamps as possible

Now we move to another variation of the problem: given a graph, tell how many lamps can be lit (if all cannot).

Authors show, that if we are given a graph in which every vertex has non-zero in-degree, then it is always possible to lit at least half of the vertices.

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It's quite easy to show: let's say that for each of the switches we toggle it with probability $\frac{1}{2}$. It's easy to see that each vertex will be lit with probability $\frac{1}{2}$, so the expected number of lit vertices is $\frac{n}{2}$.

Equivalence of directed and undirected games

We have seen that the problem isn't hard for the class of undirected graphs with loops. How does it behave when we do not require loops for each vertex. Turns out that, in a sense, it behaves very similarly to the problem on directed graphs:

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Suppose that k is the maximal number of lamps that can be lit in the game on a directed graph G. Then there is an undirected graph G' on the same vertex set, with k loops, such that exactly the same subsets of lamps can be lit.

Equivalence proof

Let's say that M is the adjacency matrix of G. Pressing a switch in vertex v is equivalent to row addition of $M[v]$ modulo 2. So all possible configurations that are achievable from 0 by pressing buttons may be represented as row space of M.

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As we know, Gaussian elimination does not influence row space, so we can perform it (+ some renumbering) to achieve 'equivalent' matrix of the form:

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Let's say that r is the size of top left identity matrix - then first $k - r$ columns of B have odd number of ones in them, and the rest has even number of ones.

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On the diagonal of $B^T B$ we have scalar products of columns of B with themselves - if number of ones was odd, then the corresponding entry on the diagonal will be one, otherwise 0. So number of ones on diagonal of the whole matrix will be $r + k - r = k$.

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So we constructed adjacency matrix of an undirected graph with k loops that has the same set of possible configurations as the original graph G.

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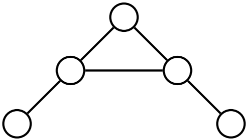

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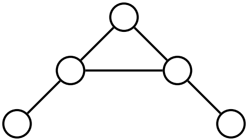

Explanation from the article (quote): *The only way to light all lamps is to press all the vertices of the triangle, and none of the other two. However, as soon as we have pressed one vertex in the triangle all its three vertices will be lit, so the next we press must be a lit vertex.*

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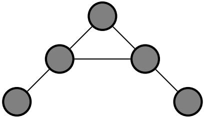

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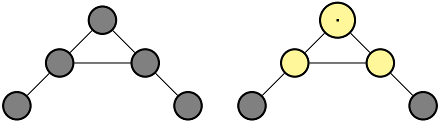

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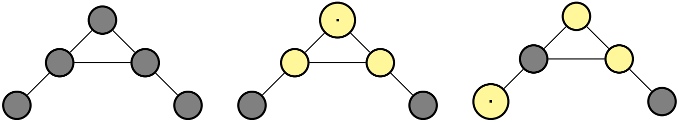

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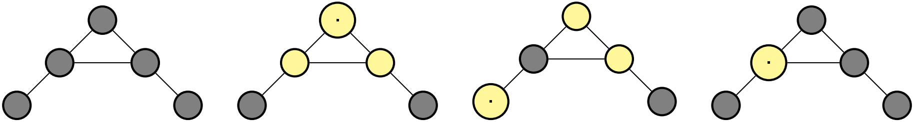

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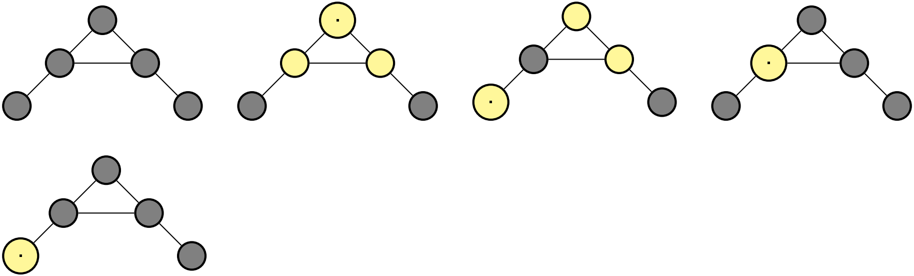

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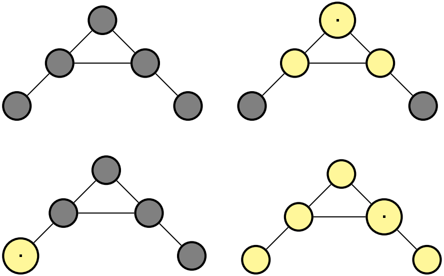

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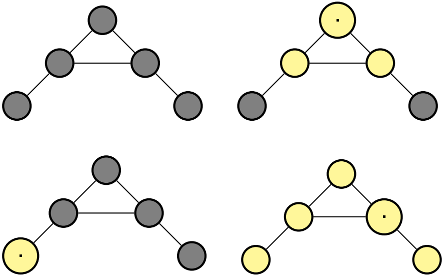

It seems that they assumed an additional constraint: each switch must be used at most once.

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We know from previous results, that in such a graph, there is a set of vertices S such that pressing each of them once will light up the whole graph. Given graph is bipartite, so we can choose two independent sets $X, Y : X \cup Y = V(G)$.

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First, we press all vertices of $X \cap S$, one by one. At each step we will use switches in dark rooms, because X is an independent set.
Then, we press $Y \cap S$ one by one - these vertices will also be dark at the time of pressing them: for $v \in Y$ there is only one vertex in Y that toggles the state of v, and it's v.

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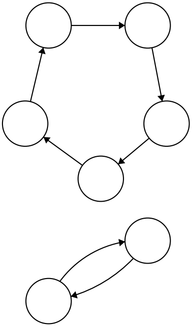
Since all entries in our matrix are ones or zeroes and we operate modulo 2, then its value can be interpreted as a number of directed circuit coverings.

Lighting any configuration

First observe that any covering that includes circuit of length at least 3 does not influence our result, as such coverings can be paired up by changing the direction of such a circuit:



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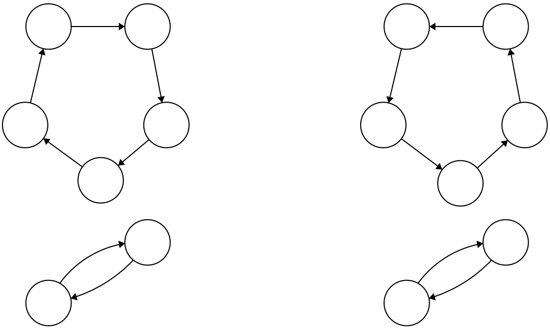

So, what counts is number of coverings with circuits of lenght 1 or 2. This represent exactly the number of complete matching, and since we count mod 2, then determinant will be non-zero iff number of complete matchings is odd.