Majority colorings of sparse digraphs

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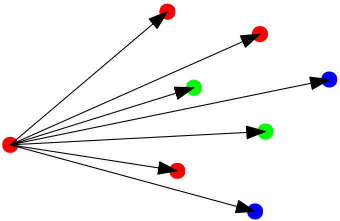


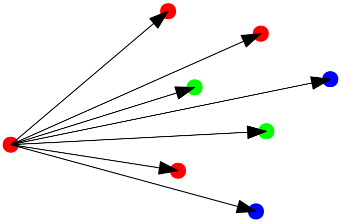
Introduction

Majority coloring

Definition

*A **majority coloring** of digraph D with k colors is an assignment $c : V(D) \to [k]$, such that*

$$\forall\, v \in V(D) \text{ \underline{at most half} of all out-neighbours of } v \text{ have the same color as } v.$$



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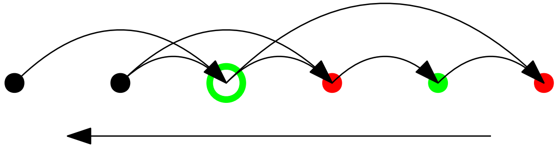
Observation

Every acyclic digraph is majority 2-colorable.



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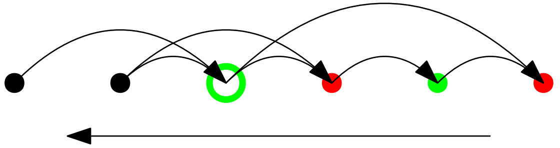
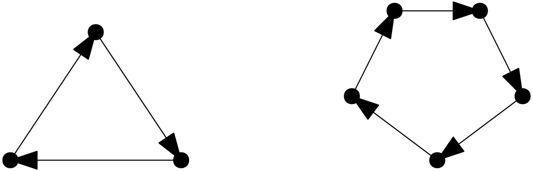


Proof.

Fix any order on the vertices, split edges into two sets by the direction they are going, color obtained graphs with two colors. Product of those colorings is a proper majority 4-coloring. □

Majority 3-colorability

There are known digraphs that are majority 3-colorable and not majority 2-colorable. The canonical examples are odd directed cycles.



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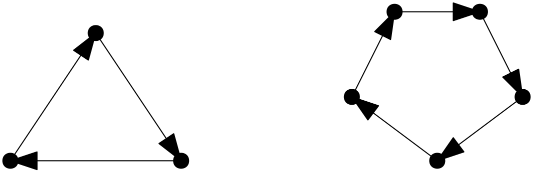

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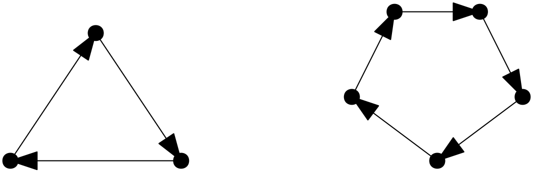

However, we don't know any digraph that is not majority 3-colorable.

Conjecture (Kreutzer et al, 2017)

Every digraph is majority 3-colorable.

What we know about majority 3-colorability?

We know that "most" digraphs are majority 3-colorable: authors of the conjecture used Lovasz Local Lemma to prove that digraphs holding certain local density conditions are majority 3-colorable:

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Theorem (Kreutzer et al, 2017)

All digraphs D satisfying

- $\delta^+(D) > 72\ln(3|V(D)|)$, *or*
- $\delta^+(D) \geq 1200$ *and* $\Delta^-(D) \leq \frac{\exp(\delta^+(D)/72)}{12\delta^+(D)}$

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Girão, Kittipassorn, and Popielarz studied tournaments in particular:

Theorem (Girão, Kittipassorn, Popielarz; 2017)

All tournaments with minimum out-degree at least 55 are majority 3 colorable.

Motivation

"All the proofs use the Local Lemma for a random coloring and hence require some **upper bound on the maximum in-degree** in terms of the minimum out-degree."

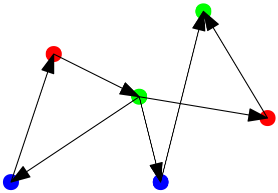
"... large maximum in-degrees seem to be outside the realm of any such probabilistic approach and it looks like it constitutes the main difficulty of the problem. This is also illustrated by the fact that **it was not even known whether planar digraphs are majority 3-colorable**"

"In this paper our main motivation is to complement the existing results on digraphs with balanced in- and out-degrees, and provide approaches for natural, broad families of digraphs, without any restriction on the maximum in-degree."

Results

Majority coloring vs chromatic number

It is easy to see that every proper coloring of digraph is also it's proper majority coloring.



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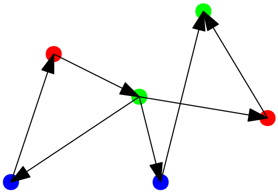

Hence every digraph D with $\chi(D) \leq 3$ is also majority 3-colorable. What about digraphs with larger chromatic number?

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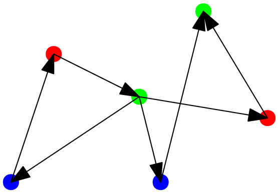


Hence every digraph D with $\chi(D) \leq 3$ is also majority 3-colorable. What about digraphs with larger chromatic number?

Theorem (M. Anastos, A. Lamaison, R. Steiner, T. Szabó; 2019)

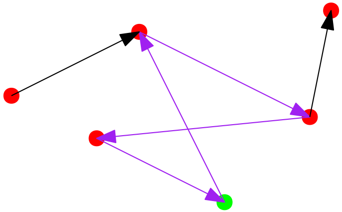
Every digraph D with $\chi(D) \leq 6$ is majority 3-colorable.

In particular, every planar digraph is majority 3-colorable.

Majority coloring vs dichromatic number

Definition

For a digraph D, its ***dichromatic number*** $\overrightarrow{\chi}(D)$ *is the smallest number of colors needed to color the vertices of D in such a way that*
there are no monochromatic cycles.



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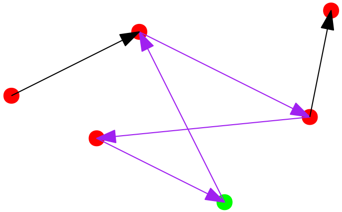


Theorem (M. Anastos, A. Lamaison, R. Steiner, T. Szabó; 2019)

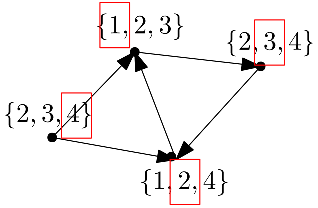
Every digraph D with $\vec{\chi}(D) \leq 3$ is majority 3-colorable.

List coloring

"For our proofs it will be crucial to work in a more general framework, involving the list coloring version of majority coloring."

Definition

*We call a digraph D **majority k-choosable** if for every k-list assignment L (i.e., assignment $L : V(D) \to 2^{\mathbb{N}}$ with $|L(v)| = k$ for every $v \in V(D)$) there is a majority coloring c satisfying $\forall_{v \in V(D)}\ c(v) \in L(v)$.*



List coloring - results

Fact

All the results about dense digraphs using the Local Lemma remain valid for majority 3-choosability.

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Theorem (M. Anastos, A. Lamaison, R. Steiner, T. Szabó; 2019)

Let D be a digraph whose underlying undirected graph is 6-choosable. Then D is majority 3-choosable.

Theorem (M. Anastos, A. Lamaison, R. Steiner, T. Szabó; 2019)

Let D be a digraph with $\overrightarrow{\chi}_l(D) \leq 3$. Then D is majority 3-choosable.

Regular graphs

Theorem (M. Anastos, A. Lamaison, R. Steiner, T. Szabó; 2019)

If $\Delta^+(D) \leq 4$ *or* $\Delta(U(D)) \leq 6$ *or* $\Delta(D) \leq 7$*, then* D *is majority 3-choosable*

Corollary

All 3*- and* 4*-regular digraphs are majority* 3*-choosable.*

Which digraphs are majority 2-colorable?

Question

Is there a characterisation of digraphs that have a majority 2-colouring (or a polynomial time algorithm to recognise such digraphs)?

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Theorem (M. Anastos, A. Lamaison, R. Steiner, T. Szabó; 2022)

Deciding whether a given digraph is majority 2-colorable is NP-complete

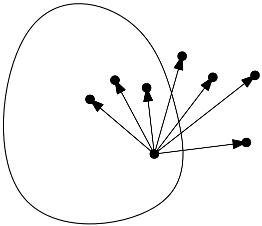
Theorem (M. Anastos, A. Lamaison, R. Steiner, T. Szabó; 2019)

If D is a digraph without odd directed cycles, then D is majority 2-choosable.

Stable sets

Definition

A set S of vertices in a digraph D is a ***stable set*** *if for each vertex $v \in S$, at most half the out-neighbours of v are also in S.*



Note that a majority coloring is a partition of vertices into stable sets.

Fractional coloring

$S(D) :=$ set of all stables sets in $V(D)$
$S(D, v) := \{T \in S(D) : v \in T\}$

Definition

*A **fractional majority coloring** is a function $f : S(D) \to \mathbb{R}_{\geq 0}$, such that*

$$\forall_{v \in V(D)} \sum_{T \in S(D,v)} f(T) \geq 1$$

*The **total weight** of a fractional majority coloring is simply $\sum_{T \in S(D)} f(T)$*

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Question

What is the minimum K such that every digraph admits a fractional majority coloring with total weight at most K?

Fractional coloring - results

Theorem (M. Anastos, A. Lamaison, R. Steiner, T. Szabó; 2019)

Every digraph D admits a fractional majority coloring with total weight at most 3.9602.

Theorem (M. Anastos, A. Lamaison, R. Steiner, T. Szabó; 2019)

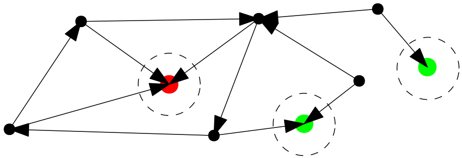
There exists a constant $C > 0$ such that for every $\epsilon > 0$ and every digraph D with $\delta^+(D) \geq C(1/\epsilon)^2 \ln(2/\epsilon)$, there exists a fractional majority coloring of D with total weight at most $2 + \epsilon$.

Proofs

Lemma 1

Lemma

Let D be a digraph which contains no odd directed cycles. Then D is majority 2-colorable. Moreover, any given pre-coloring of the sinks of D can be extended to a majority 2-coloring of D.



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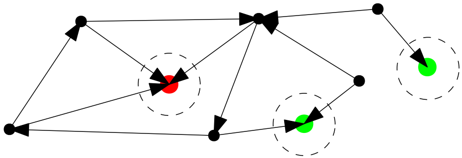


Proof sketch.

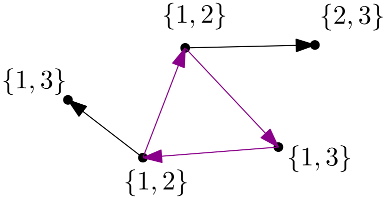
1. A digraph D contains no odd directed cycles if and only if all its strong components are bipartite.
2. Induction over the number of strong components of D.

□

Theorem 4

Theorem

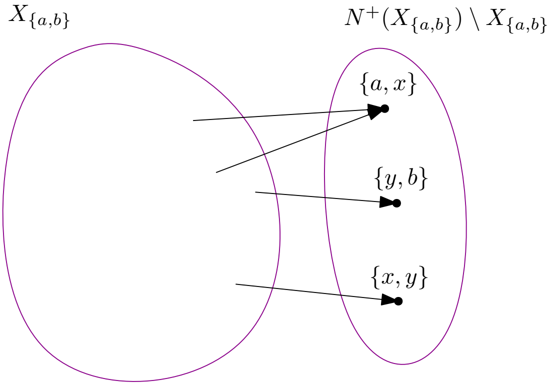
Let D be a digraph and for each $v \in V(D)$ let $L(v)$ be a list of two colors. Suppose that there exists no odd directed cycle in D all whose vertices are assigned the same list. Then there is a majority-coloring c of D such that $c(v) \in L(v)$ for all $v \in V(D)$.



Theorem 4 - proof (1/2)

$X_{\{a,b\}}$:= set of vertices with $L(v) = \{a, b\}$

$D_{\{a,b\}} := D[X_{\{a,b\}} \cup N^+(X_{\{a,b\}})]$



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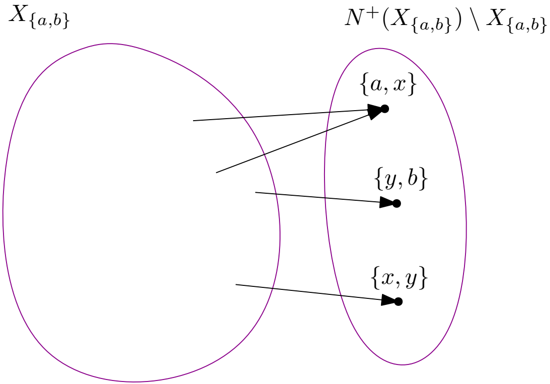
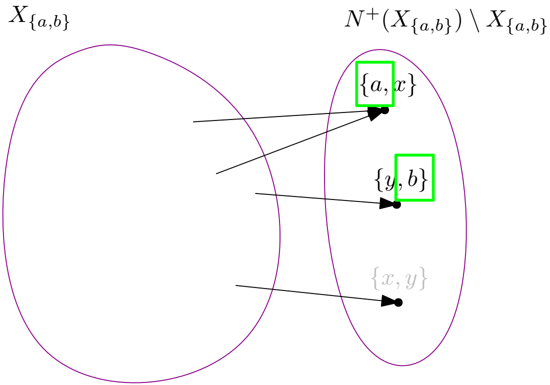


Observation 1: $D_{\{a,b\}}$ does not contain odd cycles.
Observation 2: Vertices in $N^+(X_{\{a,b\}}) \setminus X_{\{a,b\}}$ are sinks.

Theorem 4 - proof (2/2)

For each $y \in N^+(X_{\{a,b\}}) \setminus X_{\{a,b\}}$, assign to it color from $L(y) \cap \{a, b\}$ (or ignore it if it's impossible).



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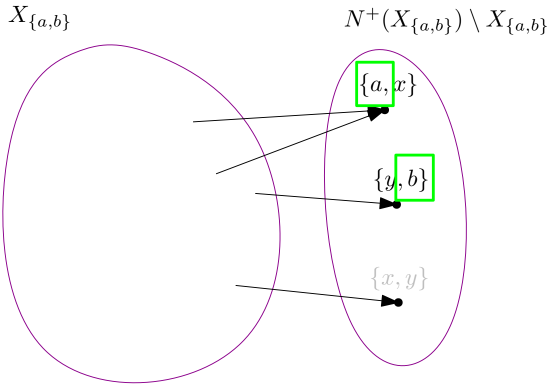

We can color remaining vertices to obtain majority coloring $c_{\{a,b\}}$ (Lemma 1). Then coloring $c(x) := c_{\{a,b\}}(x)$ for $L(x) = \{a, b\}$ is a proper majority coloring of the entire graph D.

Implications of Theorem 4

Theorem

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Let D be a digraph. Suppose there is a partition $\{X_1, X_2, X_3\}$ of the vertex set such that for every $i \in \{1, 2, 3\}$, $D[X_i]$ contains no odd directed cycles. Then D is majority 3-colorable.

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Corollary

Let D be a digraph such that $\chi(D) \leq 6$. Then D is majority 3-colorable.
Proof: $\{Y_1 \cup Y_2, Y_3 \cup Y_4, Y_5 \cup Y_6\}$

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Let D be a digraph such that $\overrightarrow{\chi}(D) \leq 3$. Then D is majority 3-colorable.

OD-3-choosability

Definition

A digraph D is ***OD-3-choosable*** *if for any assignment of color lists $L(x), x \in V(D)$ of size 3 to the vertices, there exists a choice function c (i.e. $c(x) \in L(x)$ for all $x \in V(D)$) such that no odd directed cycle in (D, c) is monochromatic.*

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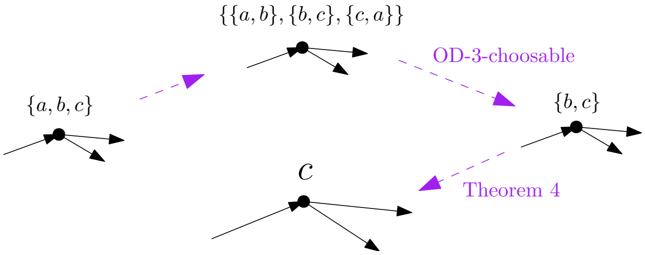
OD-3-choosability - proof



Proof.

$L :=$ given color list assignment $(\forall_{v \in V(D)} |L(v)| = 3)$
$L'(v) := \{\{C_1, C_2\} \mid C_1 \neq C_2 \in L(v)\}$ - unordered pairs of colors in $L(v)$.

D is OD-3-choosable $\implies$ there exists assignment L'', such that $L''(v) \in L'(v)$ and there is no "monochromatic" odd cycle in (D, L'').
Applying theorem 4 gives us majority coloring c, such that $c(v) \in L''(v) \subseteq L(v)$ for every vertex v. $\square$

Proof for 6-choosability

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$L_6(v) := \{C_1, C'_1, C_2, C'_2, C_3, C'_3\}$ ($C_i \in L(v)$, C'_i is a distinct copy of C_i).
D is 6-choosable $\implies \exists$ proper coloring c_6 satisfying $c_6(v) \in L_6(v)$
$c(v) := C_i$ for $c_6(v) \in \{C_i, C'_i\}$. Then $c(v) \in L(v)$ and each color class induces bipartite graph, and hence doesn't contain odd directed cycle. $\square$

List dichromatic number

Theorem

Let D be a digraph with $\vec{\chi}_l(D) \leq 3$. Then D is majority 3-choosable.

Summary

Open questions

Definition

α-majority coloring *- at most $\alpha \cdot d^+(v)$ vertices in $N^+(v)$ have the same color as v.*

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For every integer $k \geq 1$, every digraph D has a $\frac{1}{k}$-majority $(2k-1)$-coloring.

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Conjecture

For every integer $k \geq 1$, every digraph D has a $\frac{1}{k}$-majority $(2k-1)$-coloring.

- Is every 5-regular digraph $\frac{1}{3}$-majority 5-colorable?
- Does every digraph with $\chi(D) \leq 6$ have a $\frac{1}{3}$-majority 5-coloring?
- Does every digraph D with $\overrightarrow{\chi}(D) \leq 3$ have a $\frac{1}{k}$-majority $(2k-1)$-coloring for every $k \geq 1$?

Thank you!