On topological aspects of orientations

Grzegorz Gawryał

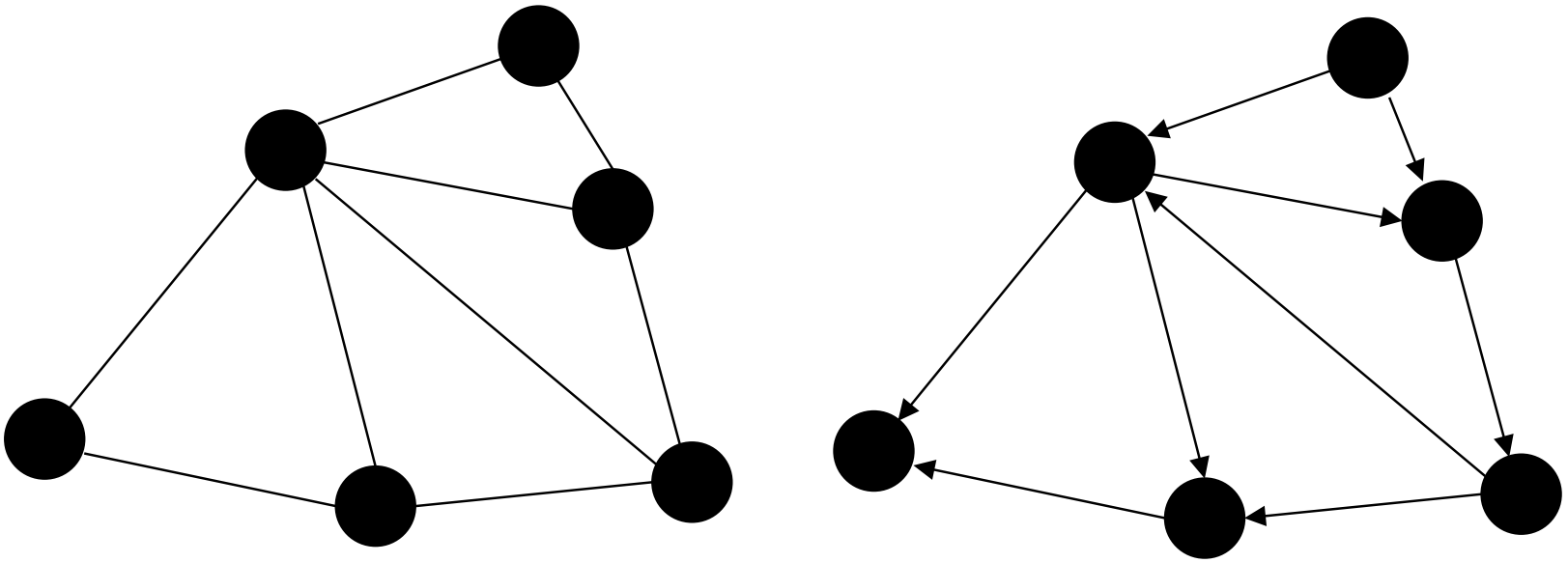
Jagiellonian University

Thursday 8 December, 2022

Based on H. de Fraysseix and P. Ossona de Mendez publication

Orientation

Orientation – any assignation of directions to each edge of a graph.



Bounded orientation existence lemma

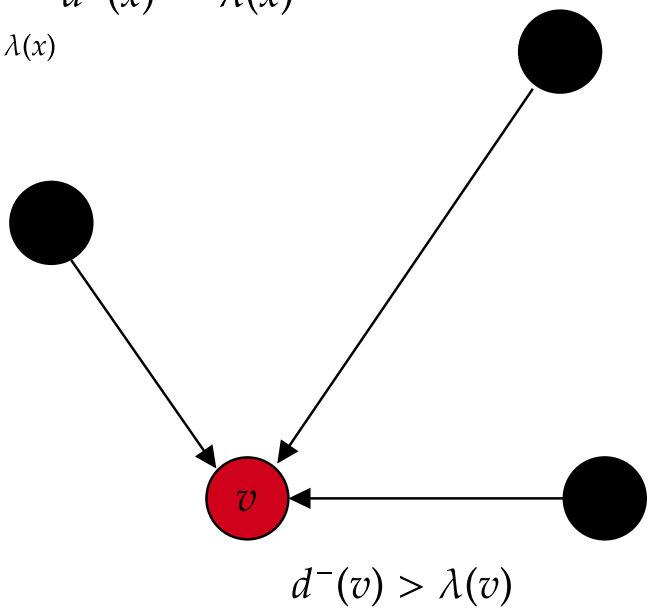
Lemma (Bounded orientation existence)

Given graph G and indegree upper bound function $\lambda : V(G) \to \mathbb{N}$, G has orientation, such that $d^-(x) \leq \lambda(x)$ iff
$\forall H \subseteq G \, \forall x \in V(H) : \sum_{x \in V(H)} \lambda(x) \geq E(H)$

Bounded orientation existence lemma - proof

For orientation minimizing

$$\sum_{x:\, d^-(x) > \lambda(x)} d^-(x) - \lambda(x)$$



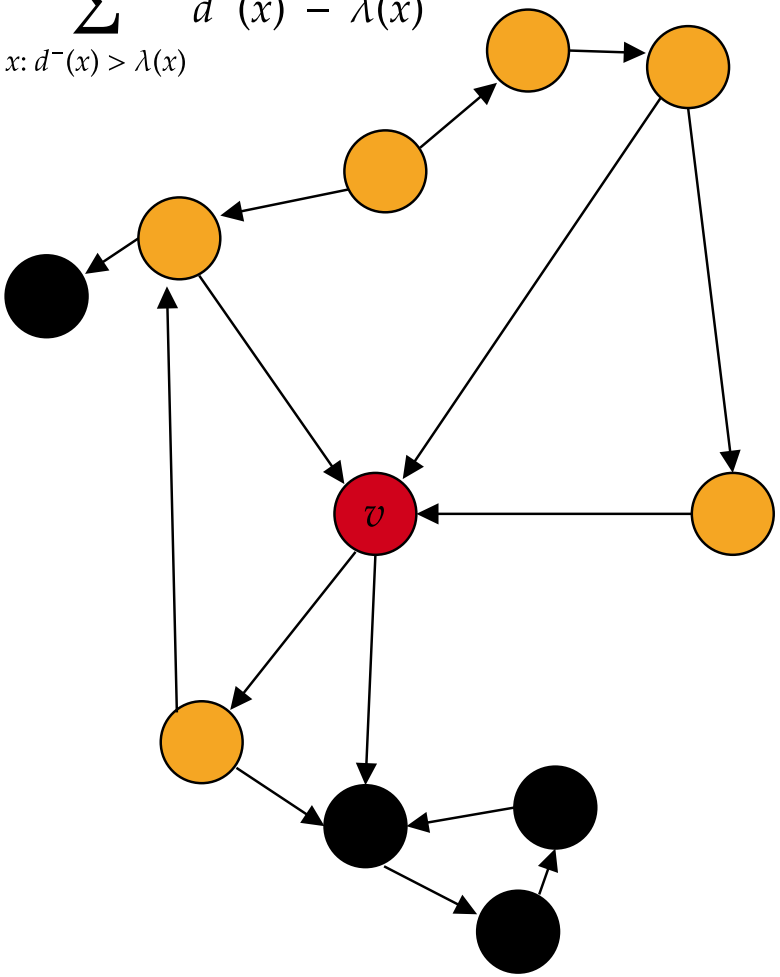
$$\forall H \subseteq G\ \forall x \in V(H): \sum_{x \in V(H)} \lambda(x) \geq E(H),$$

then G has orientation, such that $d^-(x) \leq \lambda(x)$

Bounded orientation existence lemma - proof

For orientation minimizing

$$\sum_{x:\, d^-(x) > \lambda(x)} d^-(x) - \lambda(x)$$



$$\forall H \subseteq G\ \forall x \in V(H): \sum_{x \in V(H)} \lambda(x) \geq E(H),$$

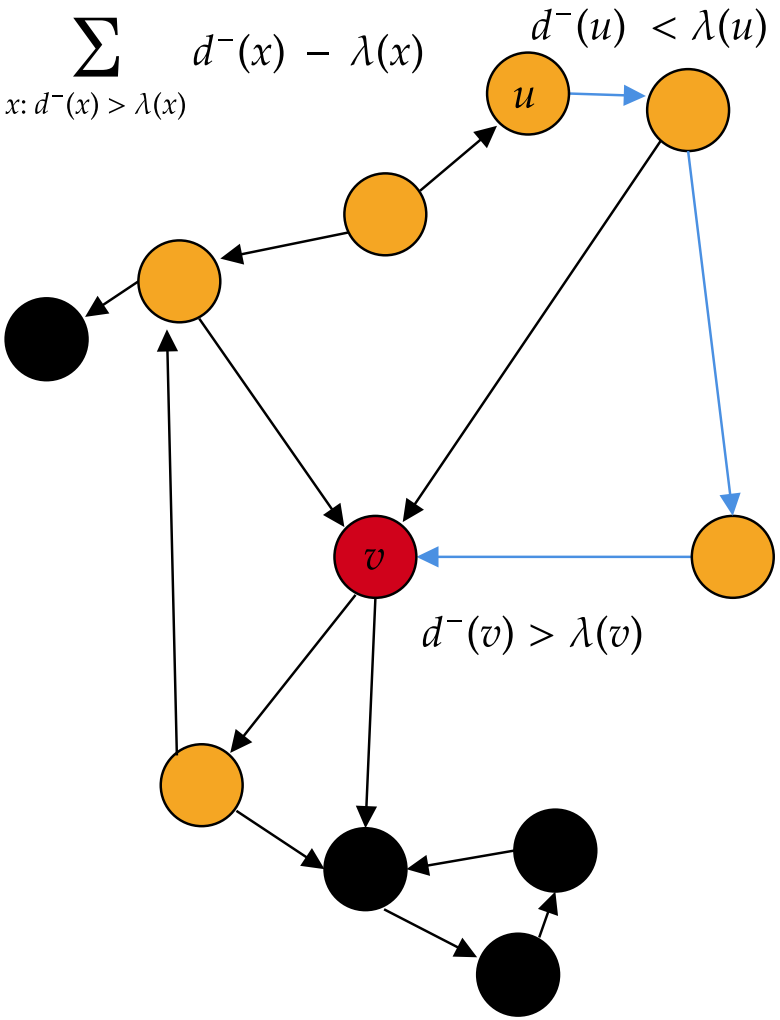
then G has orientation, such that $d^-(x) \leq \lambda(x)$

$$\sum_{x \in V(G_{\rightsquigarrow v})} \lambda(x) \geq E(G_{\rightsquigarrow v}) = \sum_{x \in V(G_{\rightsquigarrow v})} d^-(x)$$

Bounded orientation existence lemma - proof

For orientation minimizing

$$\sum_{x:\, d^-(x) > \lambda(x)} d^-(x) - \lambda(x)$$



$$\forall H \subseteq G\ \forall x \in V(H): \sum_{x \in V(H)} \lambda(x) \geq E(H),$$

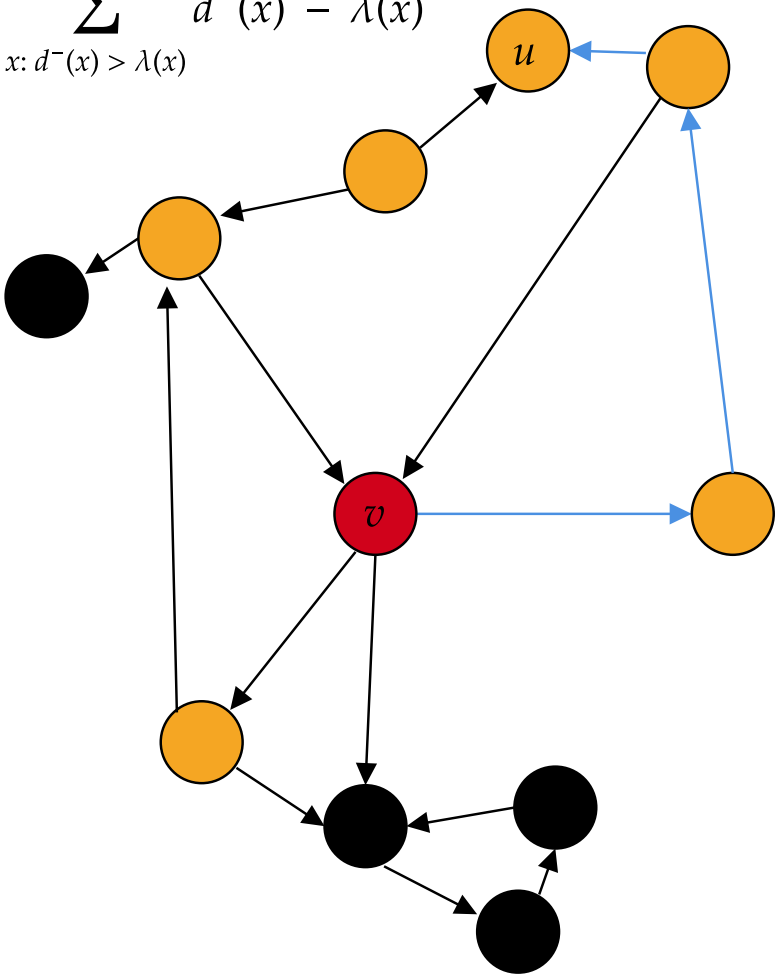
then G has orientation, such that $d^-(x) \leq \lambda(x)$

$$\sum_{x \in V(G_{\rightsquigarrow v})} \lambda(x) \geq E(G_{\rightsquigarrow v}) = \sum_{x \in V(G_{\rightsquigarrow v})} d^-(x)$$

Bounded orientation existence lemma - proof

For orientation minimizing

$$\sum_{x:\, d^-(x) > \lambda(x)} d^-(x) - \lambda(x)$$



$$\forall H \subseteq G\ \forall x \in V(H): \sum_{x \in V(H)} \lambda(x) \geq E(H),$$

then G has orientation, such that $d^-(x) \leq \lambda(x)$

$$\sum_{x \in V(G_{\rightsquigarrow v})} \lambda(x) \geq E(G_{\rightsquigarrow v}) = \sum_{x \in V(G_{\rightsquigarrow v})} d^-(x)$$

Planar graphs

- Maximal planar graphs have $3n-6$ edges, so from the previous lemma they have 3-orientation – all **internal** vertices have indegree exactly 3.

Planar graphs

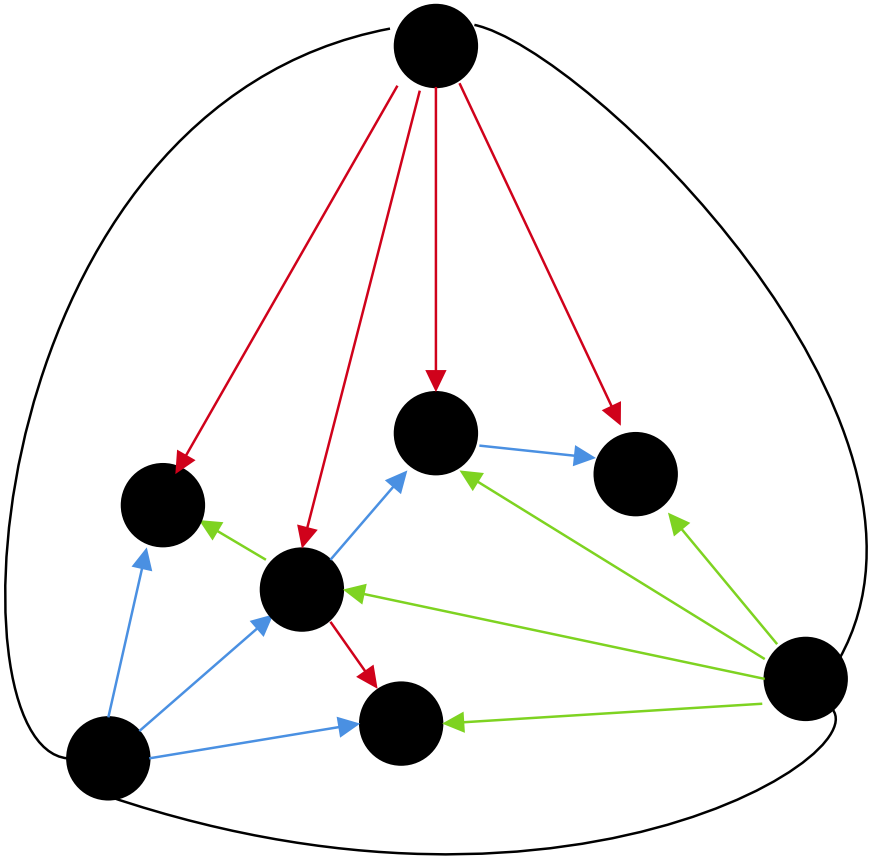
- Maximal planar graphs have $3n-6$ edges, so from the previous lemma they have 3-orientation – all **internal** vertices have indegree exactly 3.
- Similar result could be obtained by decomposing planar graph into 3 spanning trees, then orienting edges from parent to children

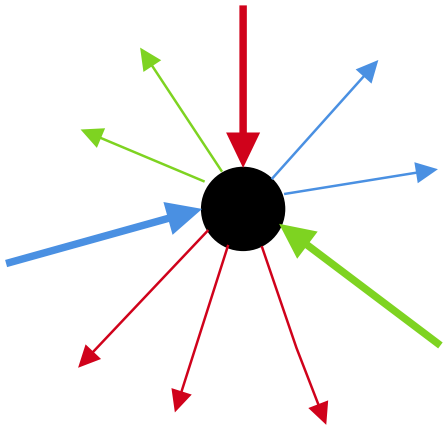
Planar graphs

- Maximal planar graphs have $3n-6$ edges, so from the previous lemma they have 3-orientation – all **internal** vertices have indegree exactly 3.
- Similar result could be obtained by decomposing planar graph into 3 spanning trees, then orienting edges from parent to children
- Actually, we'll show bijection between 3-orientations and Schnyder decomposition – special kind of decomposition into 3 spanning trees

Schnyder decomposition

Partition and orientation of internal edges into 3 spanning directed trees, rooted at r_1, r_2, r_3, preserving specific clockwise order for each vertex.

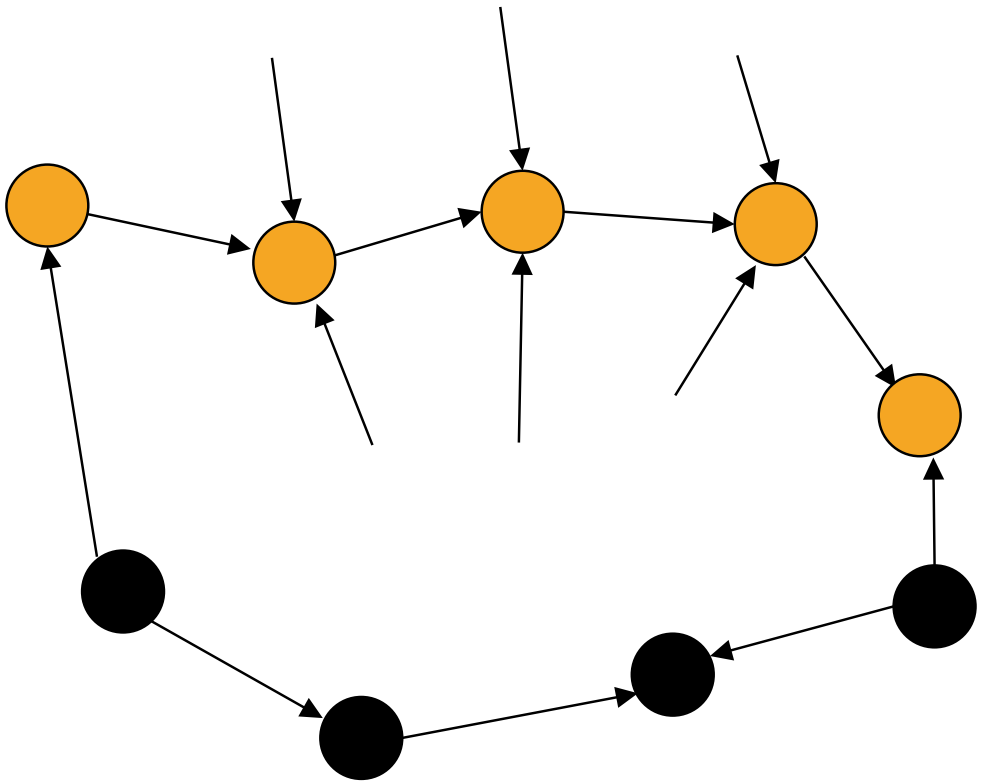




Balanced chain lemma

Definition

Balanced chain – directed path being a part of some undirected cycle, whose all vertices, except possibly endpoints, have incoming edges from inside and outside cycle.



Balanced chain lemma - proof

Lemma (Balanced chain decomposition)

No undirected cycle γ is a sum of at most 2 balanced chains.

Balanced chain lemma - proof

Lemma (Balanced chain decomposition)

No undirected cycle γ is a sum of at most 2 balanced chains.

H - induced subgraph on vertices external or on γ.

Balanced chain lemma - proof

Lemma (Balanced chain decomposition)

No undirected cycle γ is a sum of at most 2 balanced chains.

H - induced subgraph on vertices external or on γ.

$E(H) = 3V(H) - 6 - (|\gamma| - 3) = 3V(H) - |\gamma| - 3$

Balanced chain lemma - proof

Lemma (Balanced chain decomposition)

No undirected cycle γ is a sum of at most 2 balanced chains.

H - induced subgraph on vertices external or on γ.

$E(H) = 3V(H) - 6 - (|\gamma| - 3) = 3V(H) - |\gamma| - 3$

$\sum_{v \in V(H)} d_G^-(v) = 3V(H) - 6$

Balanced chain lemma - proof

Lemma (Balanced chain decomposition)

No undirected cycle γ is a sum of at most 2 balanced chains.

H - induced subgraph on vertices external or on γ.

$E(H) = 3V(H) - 6 - (|\gamma| - 3) = 3V(H) - |\gamma| - 3$

$\sum_{v \in V(H)} d_G^-(v) = 3V(H) - 6$

Thus, H has $3V(H) - 6 - (3V(H) - |\gamma| - 3) = |\gamma| - 3$ incoming edges from inside.

Balanced chain lemma - proof

Lemma (Balanced chain decomposition)

No undirected cycle γ is a sum of at most 2 balanced chains.

H - induced subgraph on vertices external or on γ.

$E(H) = 3V(H) - 6 - (|\gamma| - 3) = 3V(H) - |\gamma| - 3$

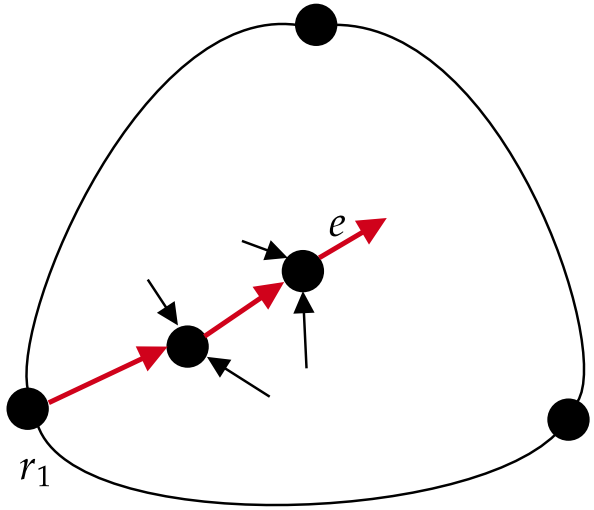
$\sum_{v \in V(H)} d_G^-(v) = 3V(H) - 6$

Thus, H has $3V(H) - 6 - (3V(H) - |\gamma| - 3) = |\gamma| - 3$ incoming edges from inside.

At least 3 vertices in γ have no incoming edge from inside.

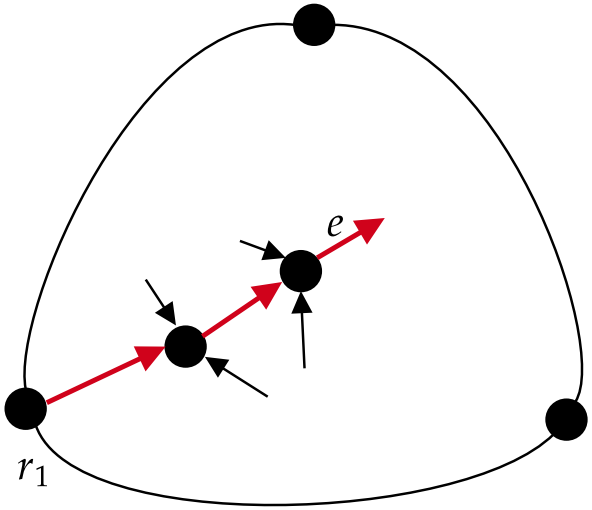
3 orientation $\rightarrow$ Schnyder decomposition

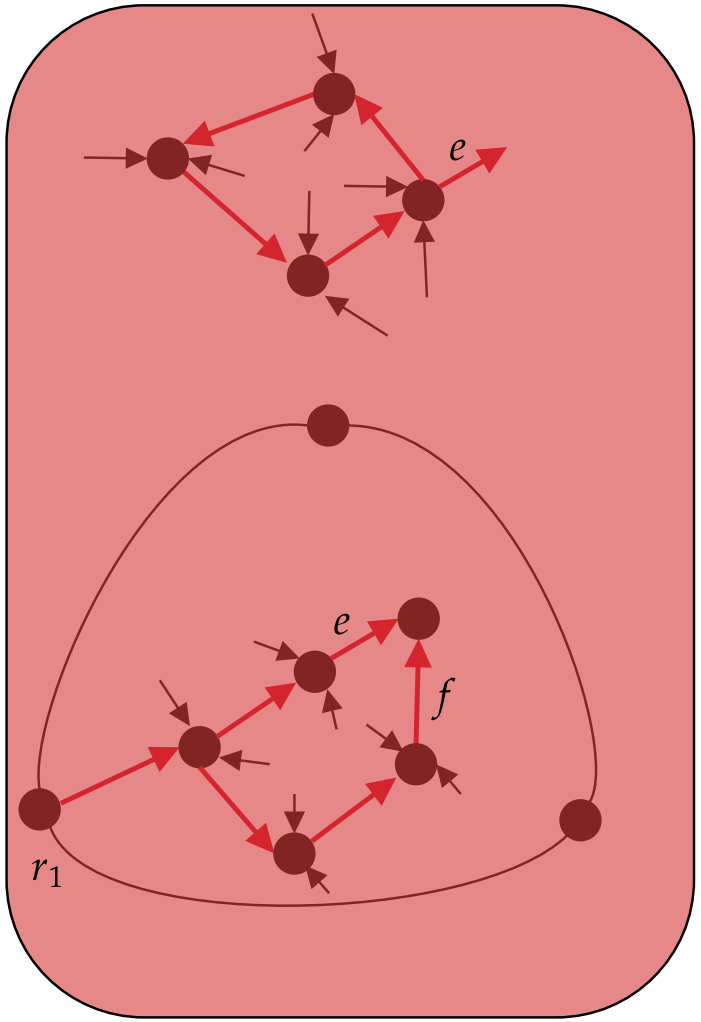
For each e backtrack using "middle" edge until r_1, r_2 or r_3 and color depending on where ended.



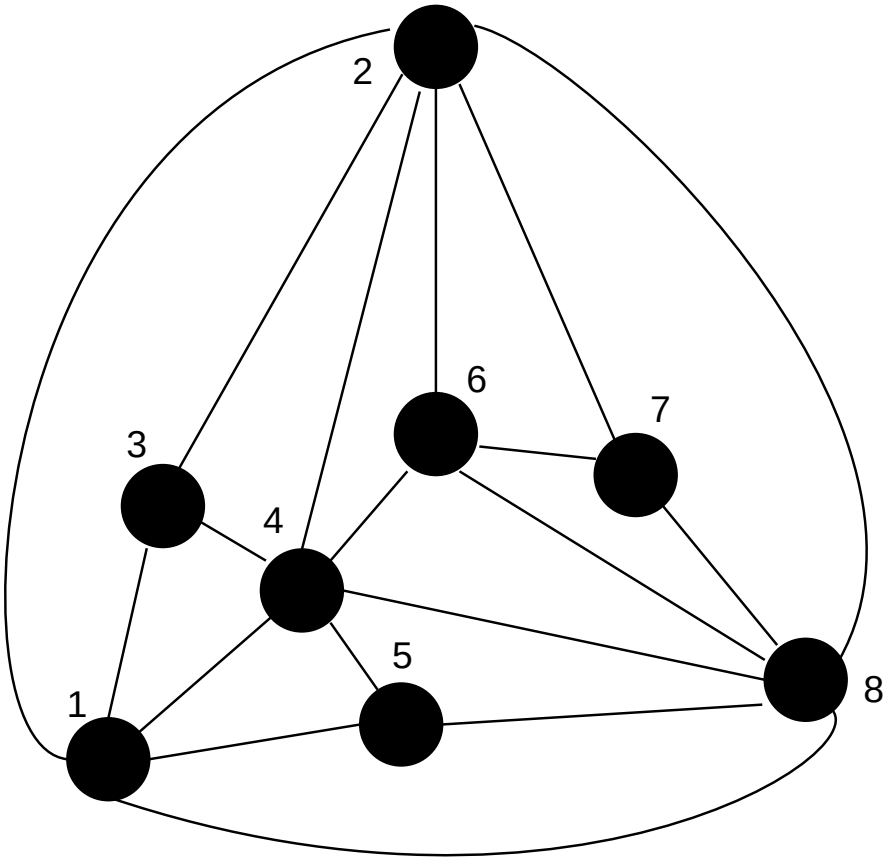
3 orientation $\rightarrow$ Schnyder decomposition

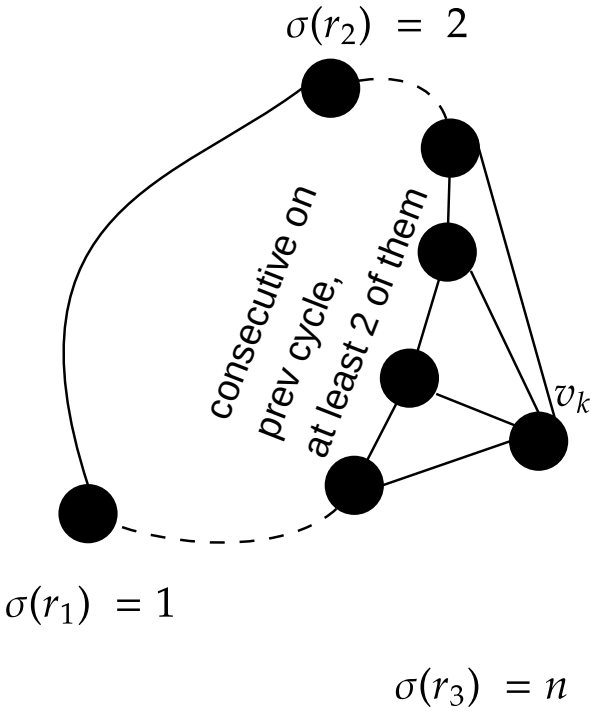
For each e backtrack using "middle" edge until r_1, r_2 or r_3 and color depending on where ended.



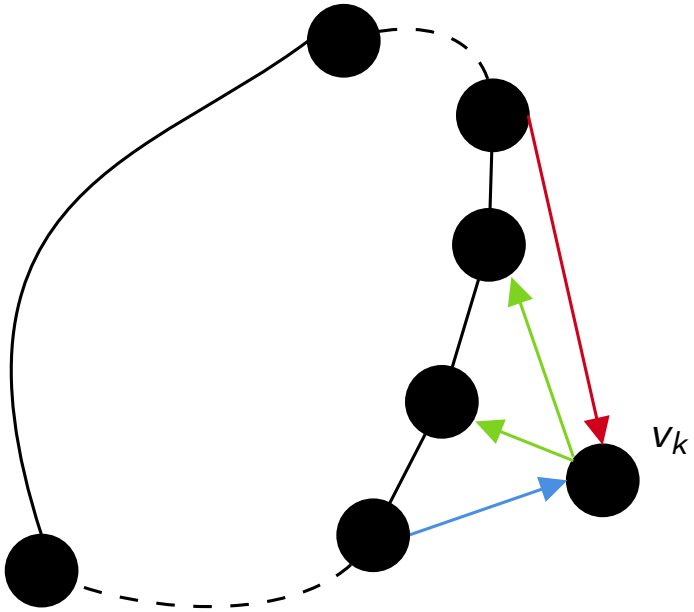


Shelling order



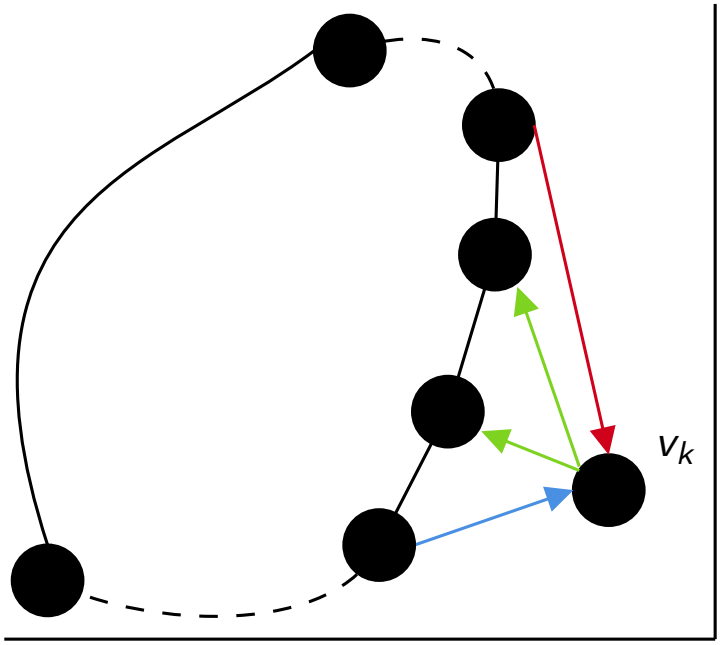


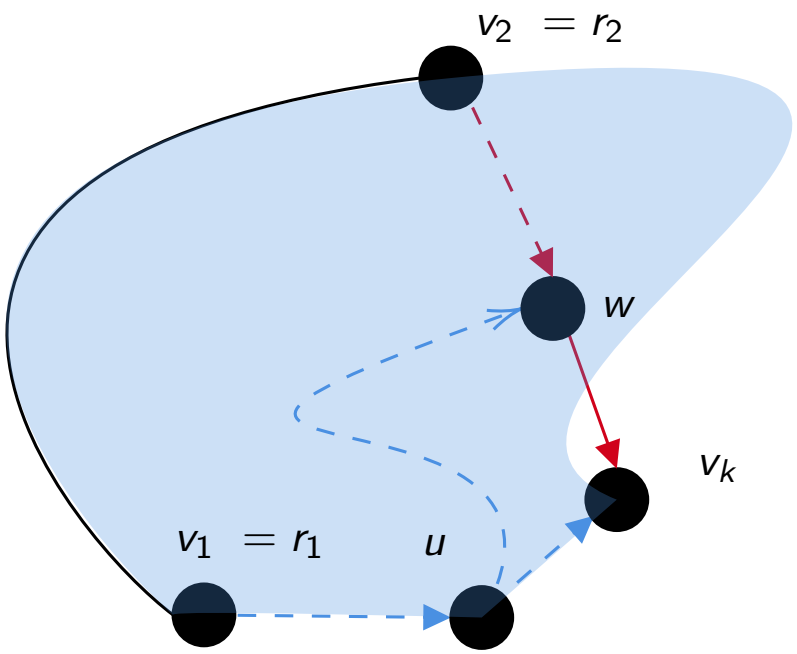
Shelling order $\to$ Schnyder decomposition



$\to$ Schnyder decomposition = as in the picture

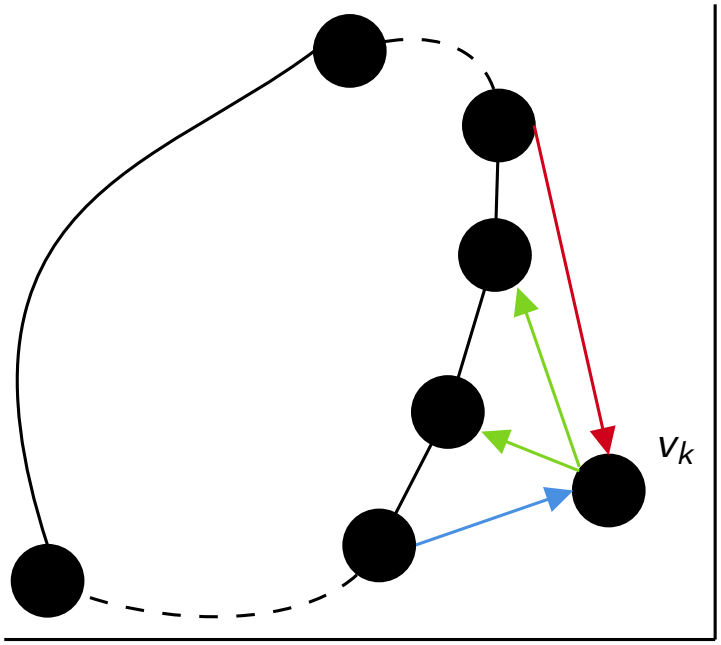
Shelling order ← Schnyder decomposition

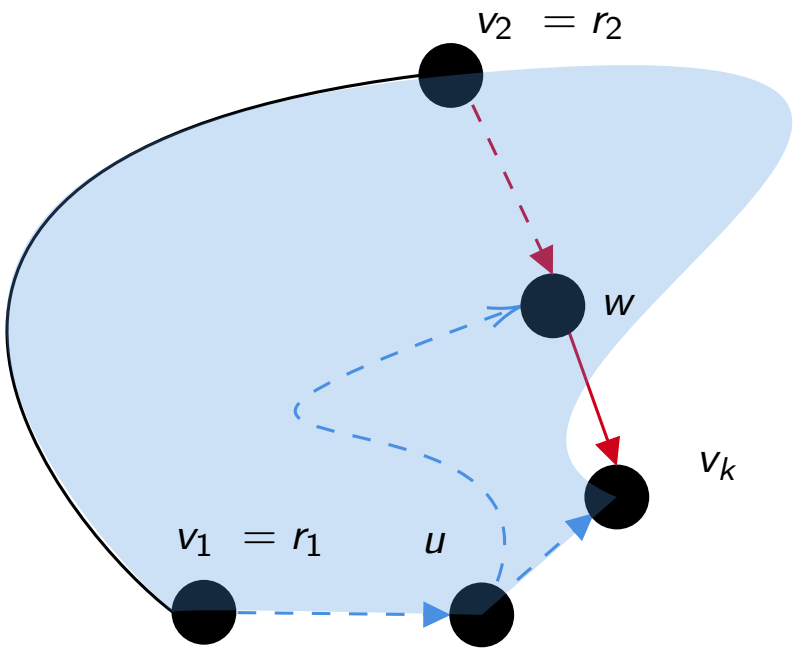




← shelling order = clockwise preorder of tree from r_1

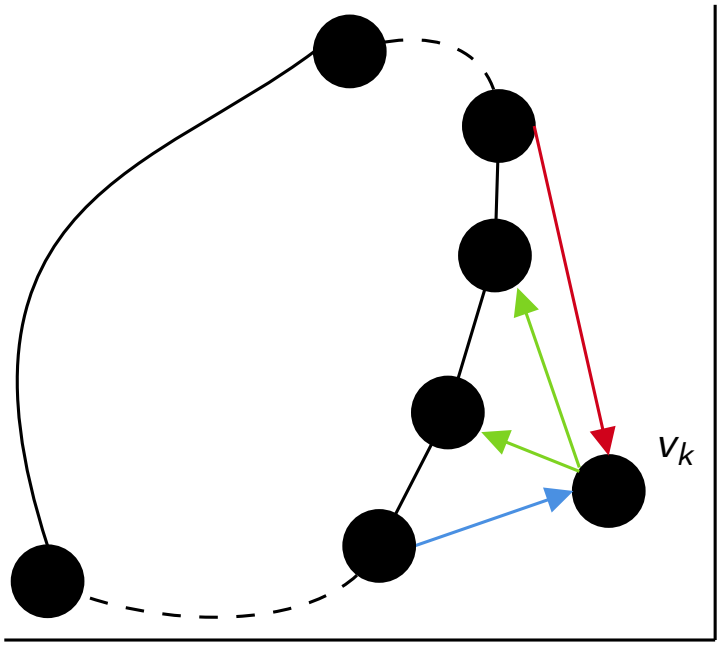
Shelling order $\leftarrow$ Schnyder decomposition

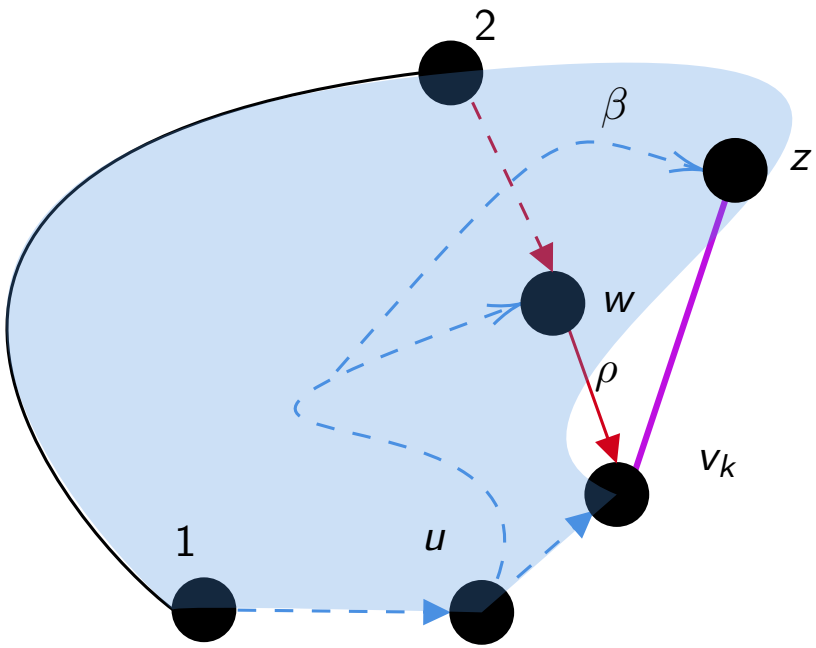




1. v_k has at least 2 neighbours in external cycle of subraph on $v_1, \ldots, v_{k-1}$:
- paths $r_1 \rightarrow v_k$ and $r_2 \rightarrow v_k$ are disjoint - balanced chain decomposition lemma
- w is visited before v_k in clockwise preorder order from r_1

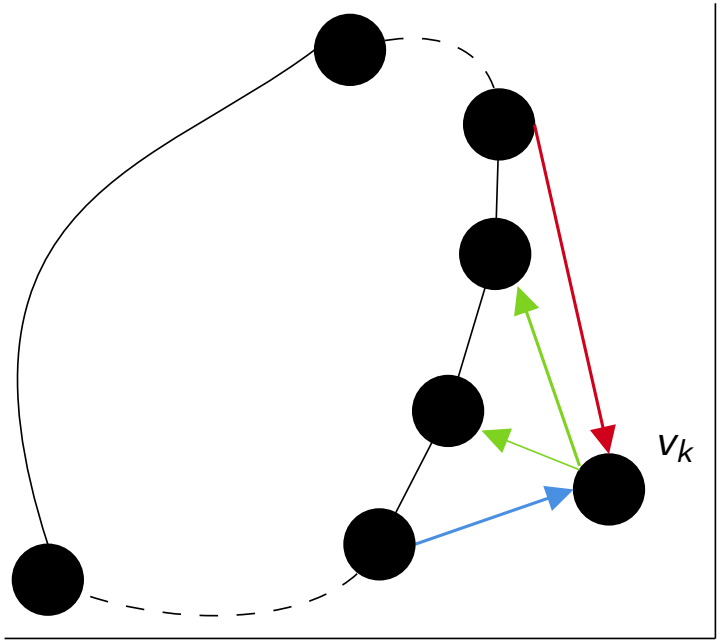
Shelling order ← Schnyder decomposition

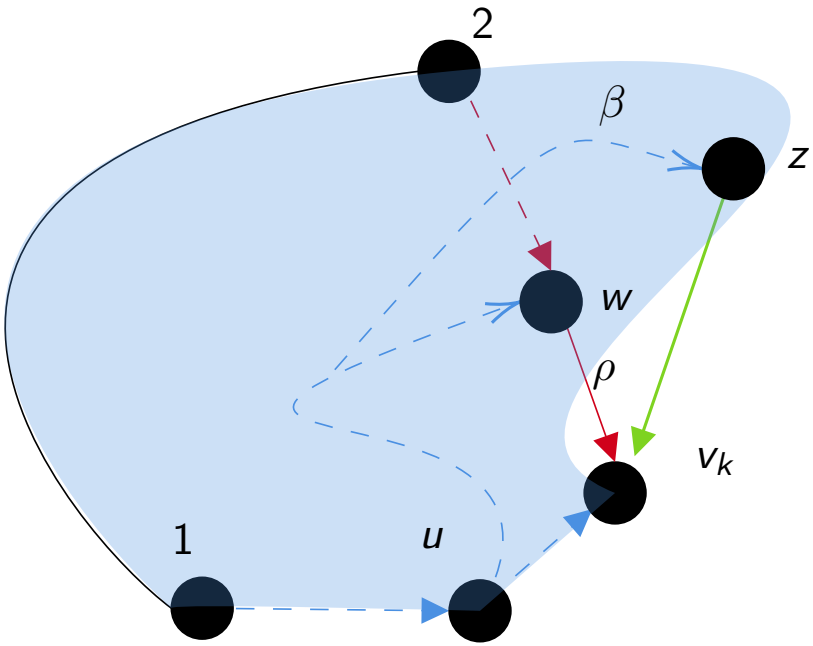




2. Red incoming edge is the rightmost looking from v_k:
- no blue edge - would create a cycle
- no green outgoing - such edges must be between incoming blue and red

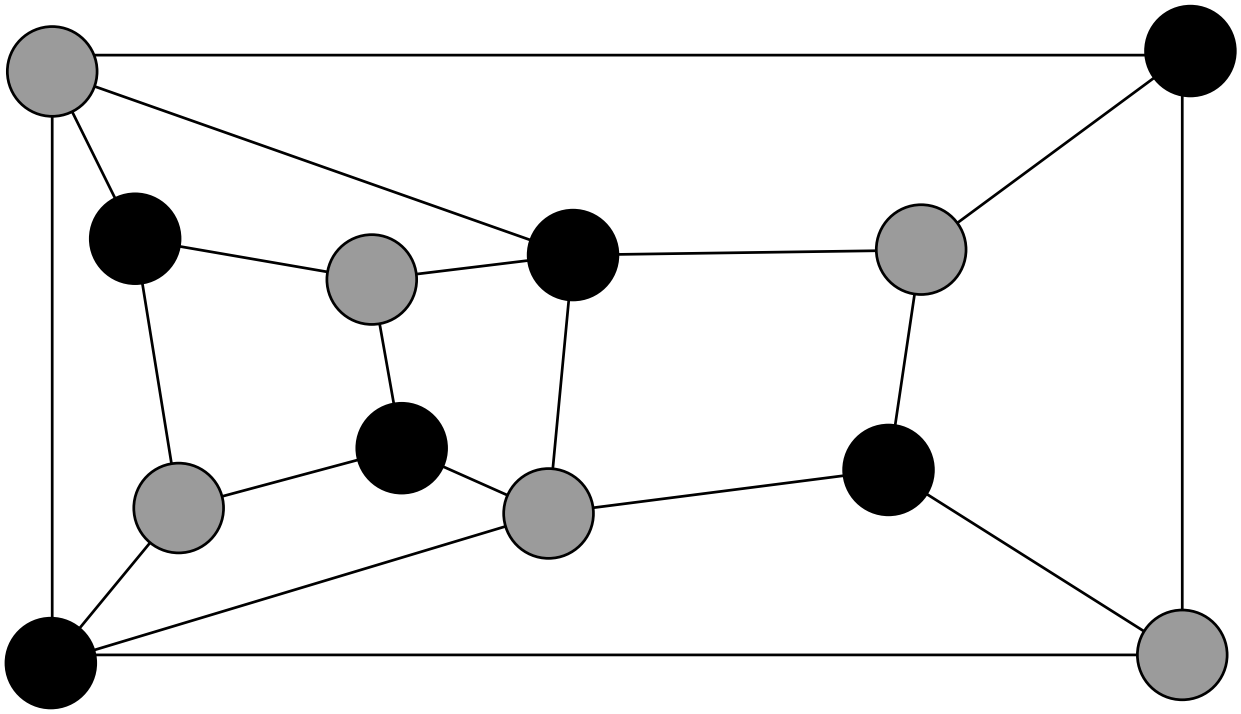
Shelling order ← Schnyder decomposition





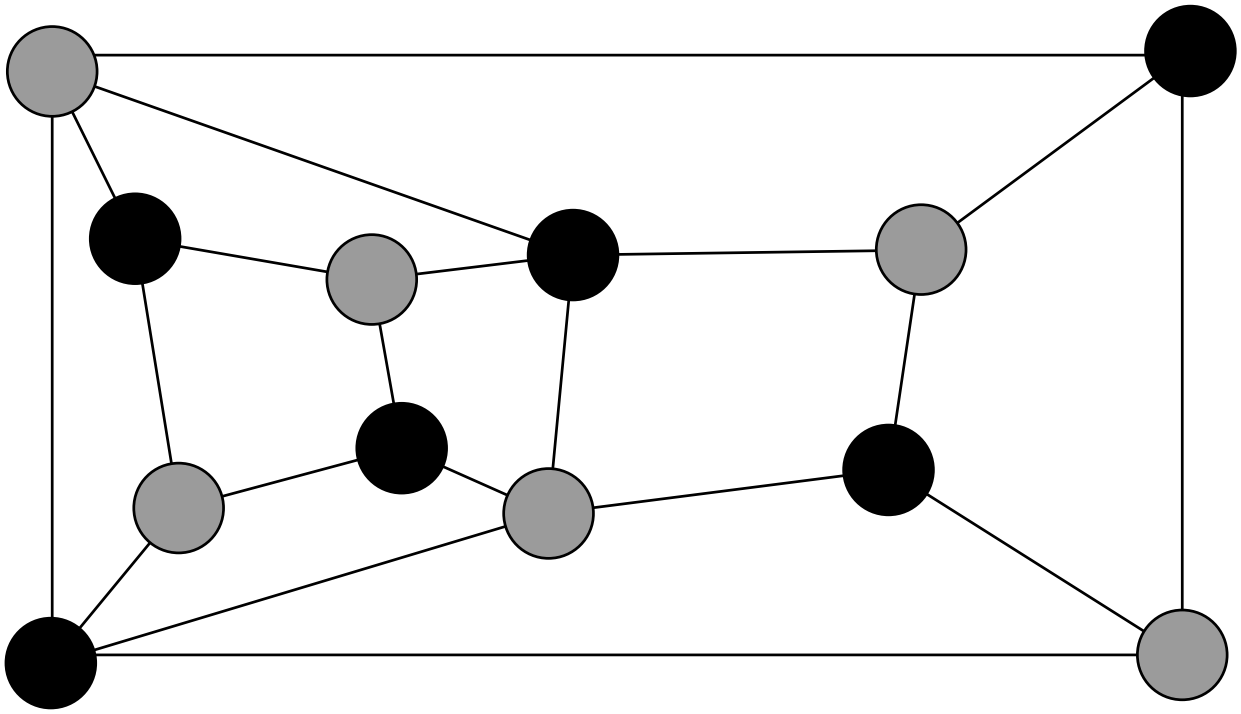
2. Red incoming edge is the rightmost looking from v_k:
   - green incoming / red outgoing - then green incoming to z would come from area between β and ρ
   - then green path to z either crosses β or ρ

Maximal bipartite planar graphs



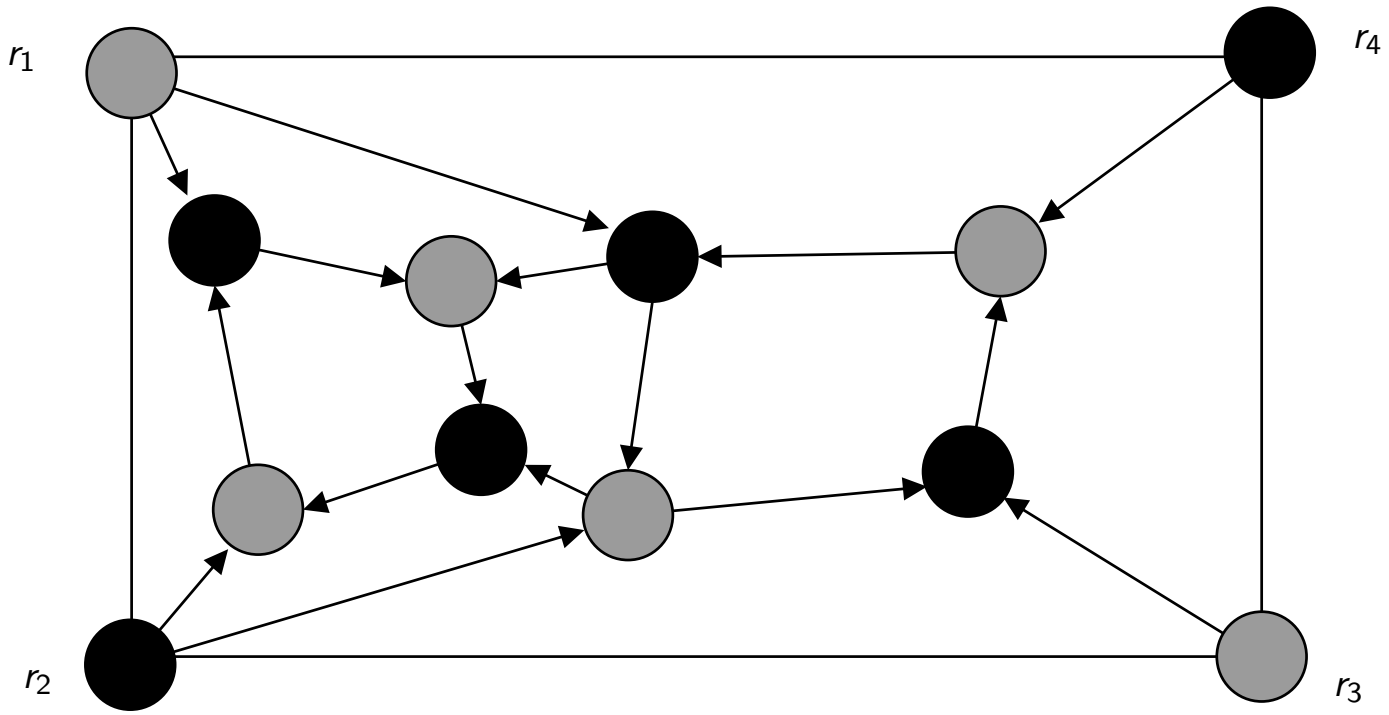
- Faces = quadrangles

Maximal bipartite planar graphs



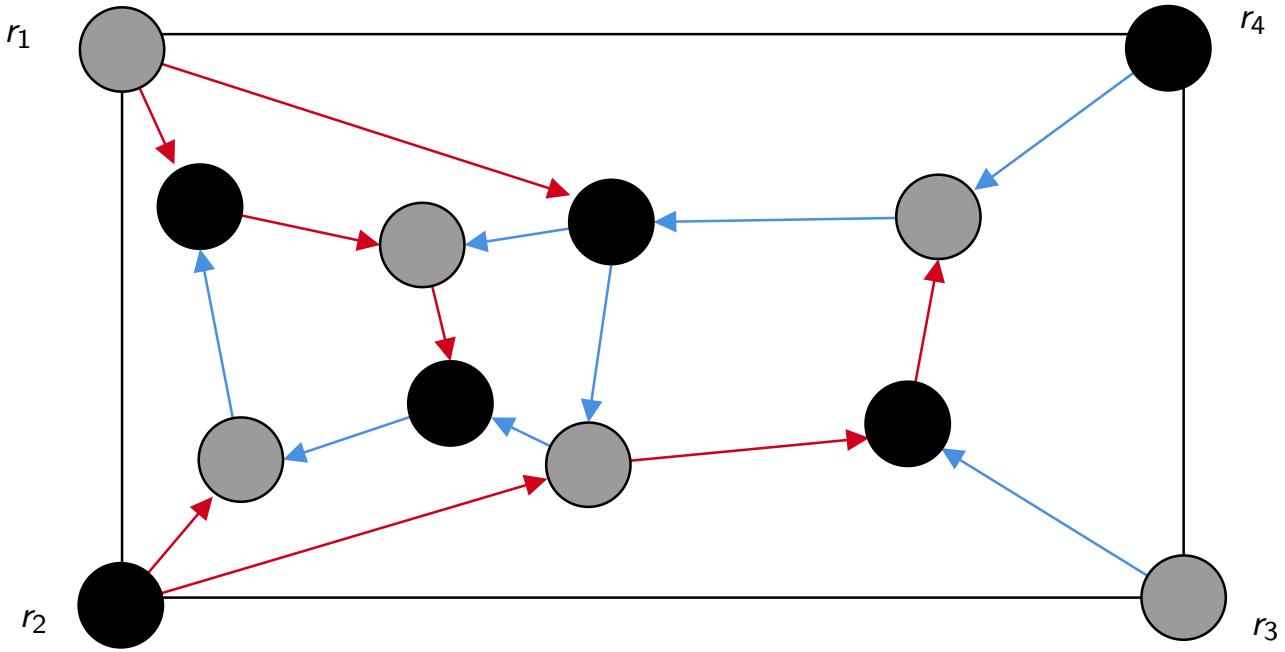
- Faces = quadrangles
- $E(H) = 2|V(H)| - 4$

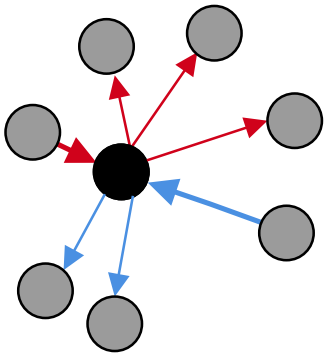
Maximal bipartite planar graphs

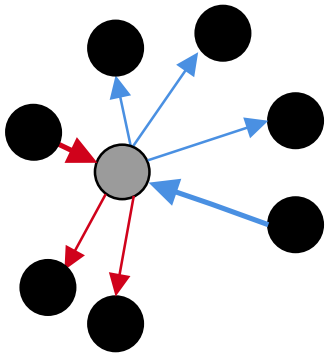


- Faces = quadrangles
- $E(H) = 2|V(H)| - 4$
- from Bounded orientation existence lemma, there exists 2-orientation – all internal vertices have indegree exactly 2.

Analogs for maximal bipartite planar graphs







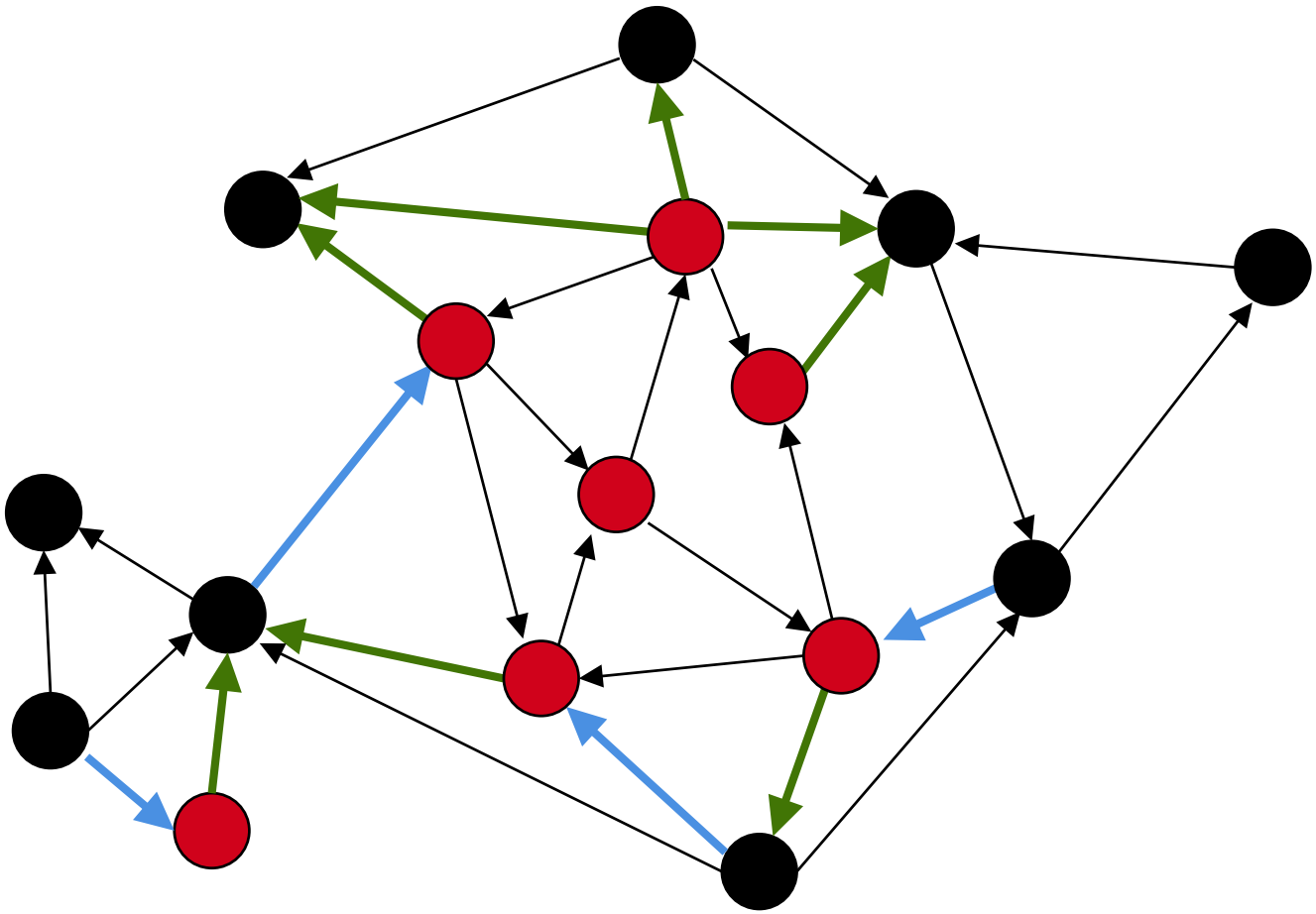
Analogs for maximal bipartite planar graphs

- Proof of bijection between separating decompositions and 2-orientations is very similar, using half-balanced chains

Analogs for maximal bipartite planar graphs

- Proof of bijection between separating decompositions and 2-orientations is very similar, using half-balanced chains
- Authors don't mention any shelling order analog

Application - connectivity - prerequisites



cocycle $\omega(X) \;=\; (\omega^+(X),\ \omega^-(X))$, where
$\omega^+(X) \;=\; \{(u \to v) \in E\colon\ u \in X,\ v \notin X\}$ $\qquad$ $\omega^-(X) \;=\; \{(v \to u) \in E\colon\ u \in X,\ v \notin X\}$

Application - connectivity- prerequisites

- Cocycle is elementary, if both X and $\overline{X}$ are weakly connected

Application - connectivity- prerequisites

- Cocycle is elementary, if both X and $\overline{X}$ are weakly connected
- Positive circuit = cocycle, where $\omega^-(X) = \emptyset$

Application - connectivity- prerequisites

- Cocycle is elementary, if both X and $\overline{X}$ are weakly connected
- Positive circuit = cocycle, where $\omega^-(X) = \emptyset$

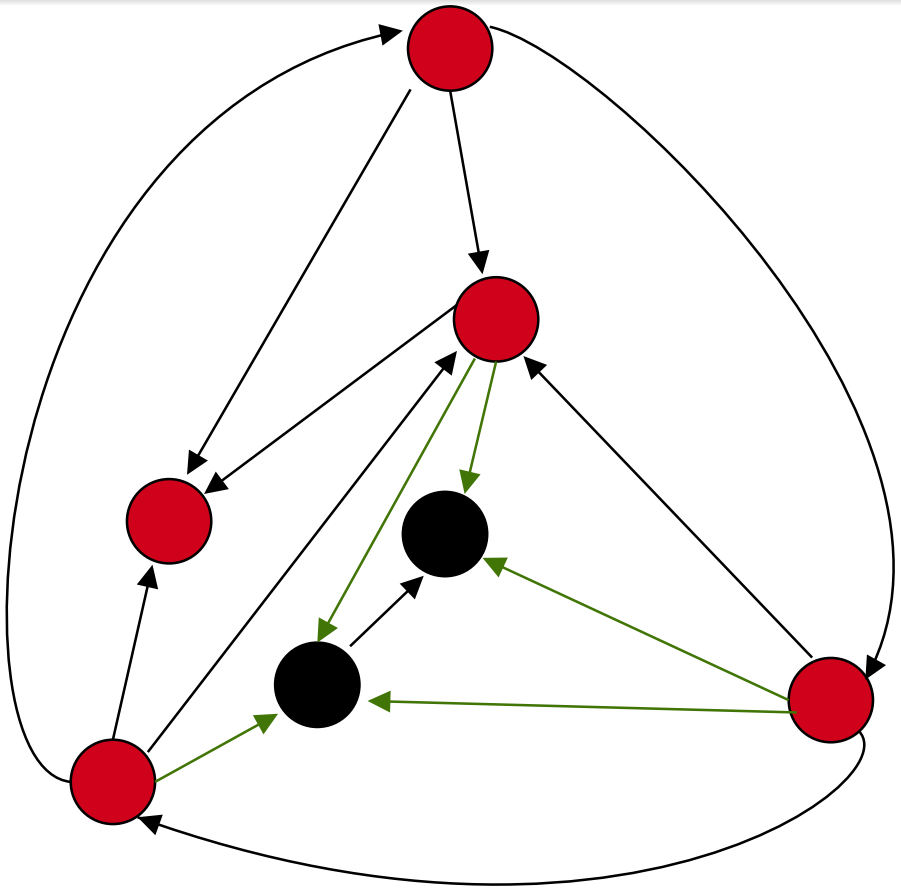
Lemma

For any 3-oriented maximal planar graph G having its external face oriented as a circuit, there exists a bijection between elementary positive cocircuits of G and undirected triangles of G.

Application - connectivity - positive cocircuit lemma

Lemma

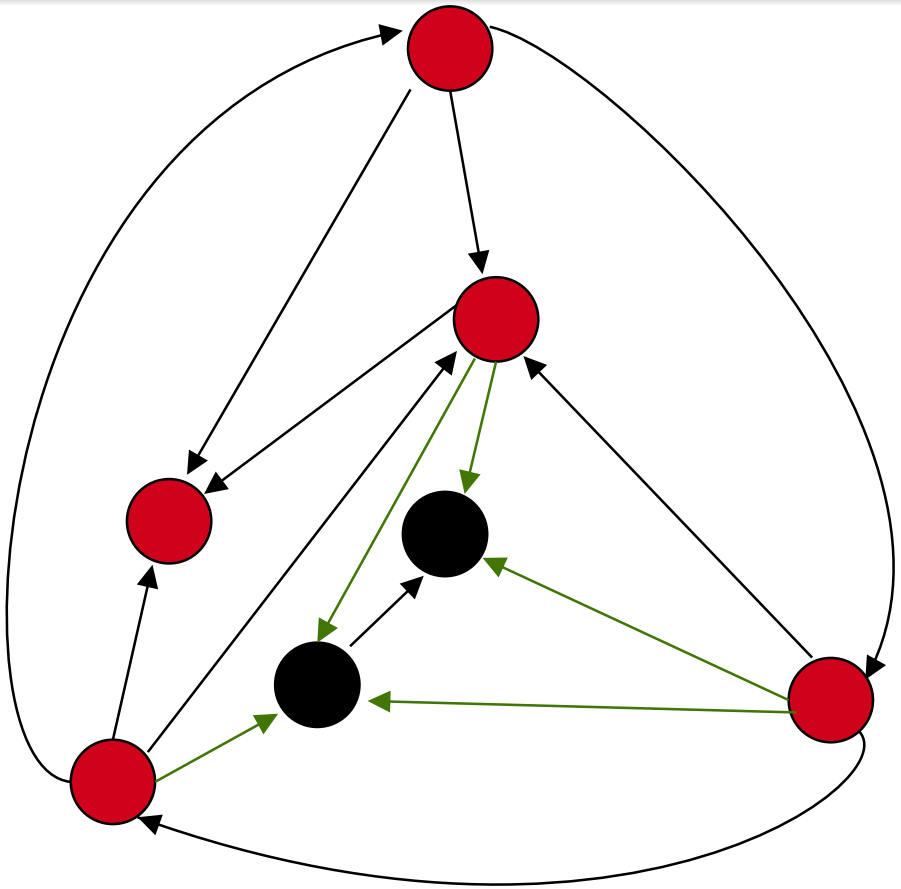
For any 3-oriented maximal planar graph G having its external face oriented as a circuit, there exists a bijection between elementary positive cocircuits of G and undirected triangles of G.



Application - connectivity - positive cocircuit lemma

Lemma

For any 3-oriented maximal planar graph G having its external face oriented as a circuit, there exists a bijection between elementary positive cocircuits of G and undirected triangles of G.

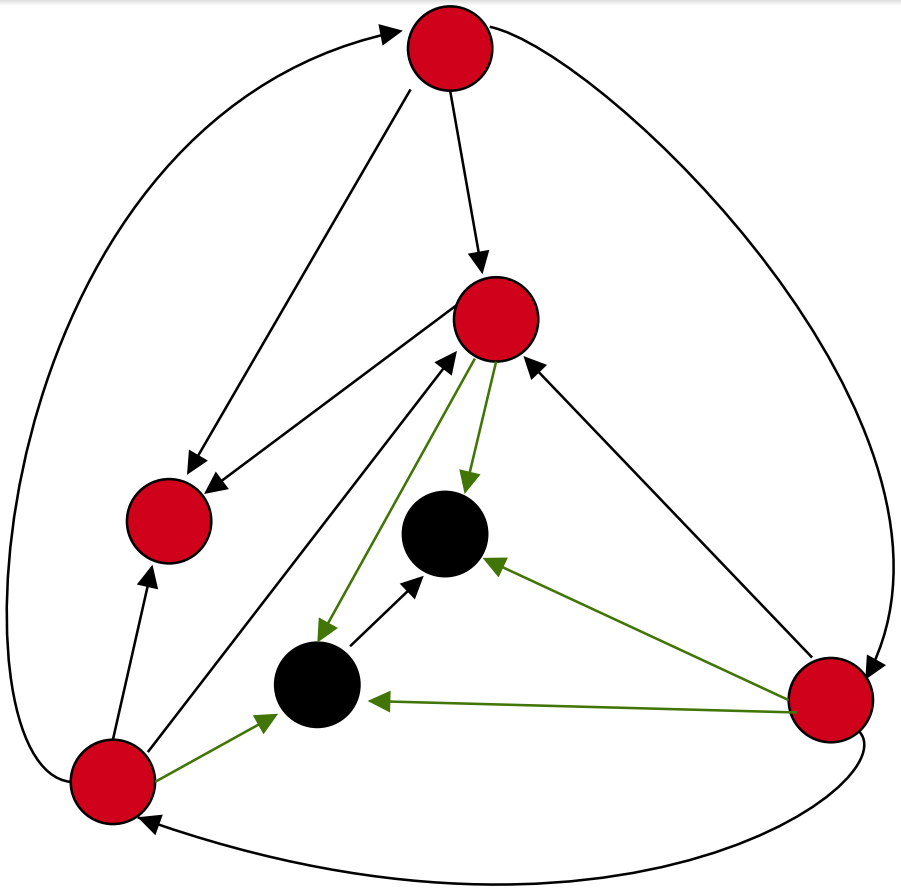


$\vec{\omega}(X)$ = elementary positive cocircuit
either X has all r_1, r_2, r_3 or none

Application - connectivity - positive cocircuit lemma

Lemma

For any 3-oriented maximal planar graph G having its external face oriented as a circuit, there exists a bijection between elementary positive cocircuits of G and undirected triangles of G.

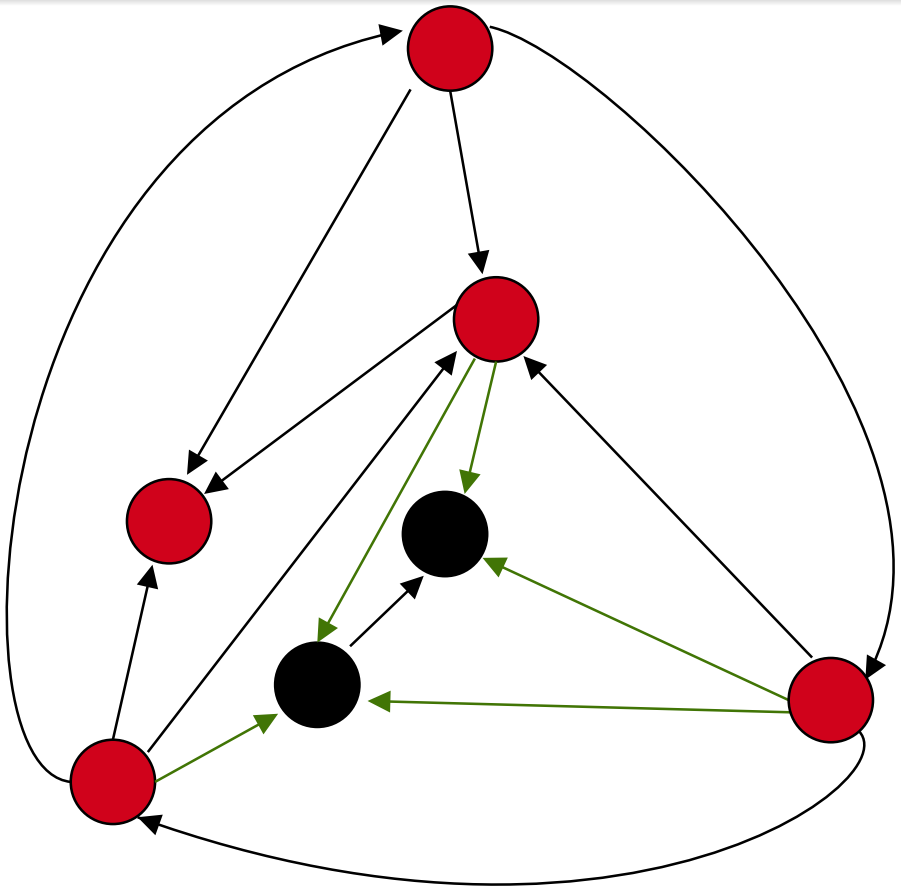


$\rightarrow$
$\omega(X)$ = elementary positive cocircuit
either X has all r_1, r_2, r_3 or none
if it had none, then
$3|X| = \sum_{v \in X} d^-_G(v) = \sum_{v \in X} d^-_{G_X}(v) = E(G_X)$,
which is impossible

Application - connectivity - positive cocircuit lemma

Lemma

For any 3-oriented maximal planar graph G having its external face oriented as a circuit, there exists a bijection between elementary positive cocircuits of G and undirected triangles of G.

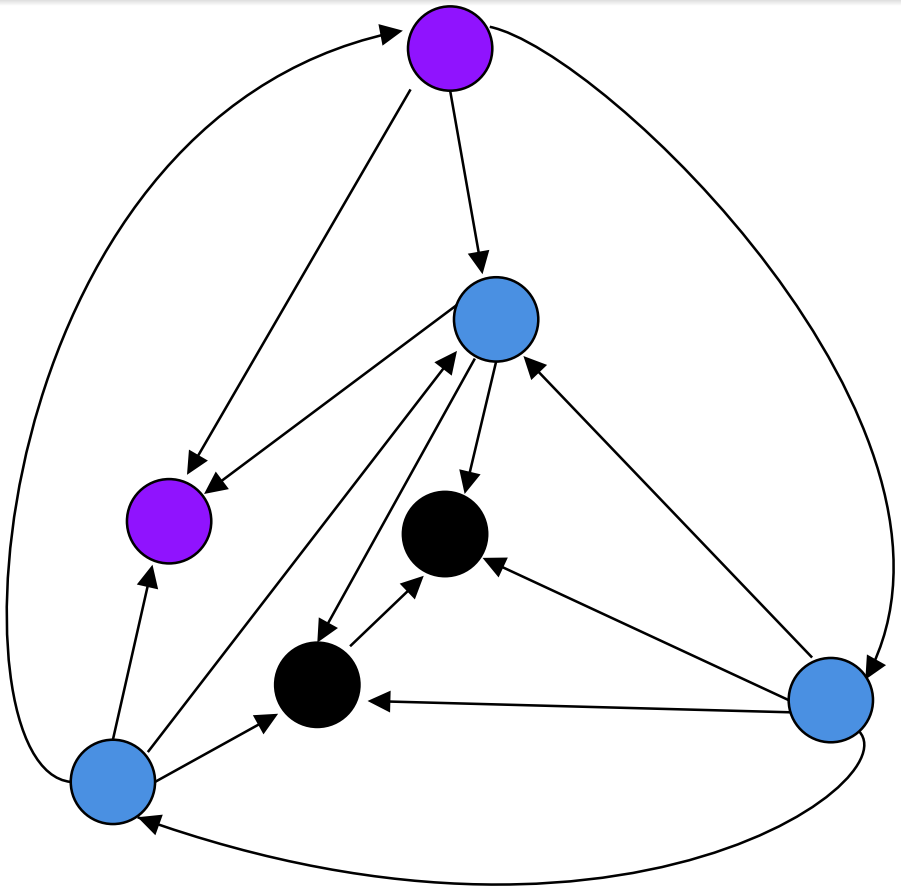


$\rightarrow$
$\omega(X) \;=\;$ elementary positive cocircuit
either X has all $r_1,\ r_2,\ r_3$ or none
if it had none, then :
$3|X| = \sum_{v\ \in X} d^-_G(v) = \sum_{v\ \in X} d^-_{G_X}(v) = \;E(G_X)$,
and if it contains each of $r_1,\ r_2,\ r_3$:
$3|X| - 6 = \;E(G_X)$
thus all faces are triangles

Application - connectivity - positive cocircuit lemma

Lemma

For any 3-oriented maximal planar graph G having its external face oriented as a circuit, there exists a bijection between elementary positive cocircuits of G and undirected triangles of G.



$\leftarrow$

T = non empty triangle

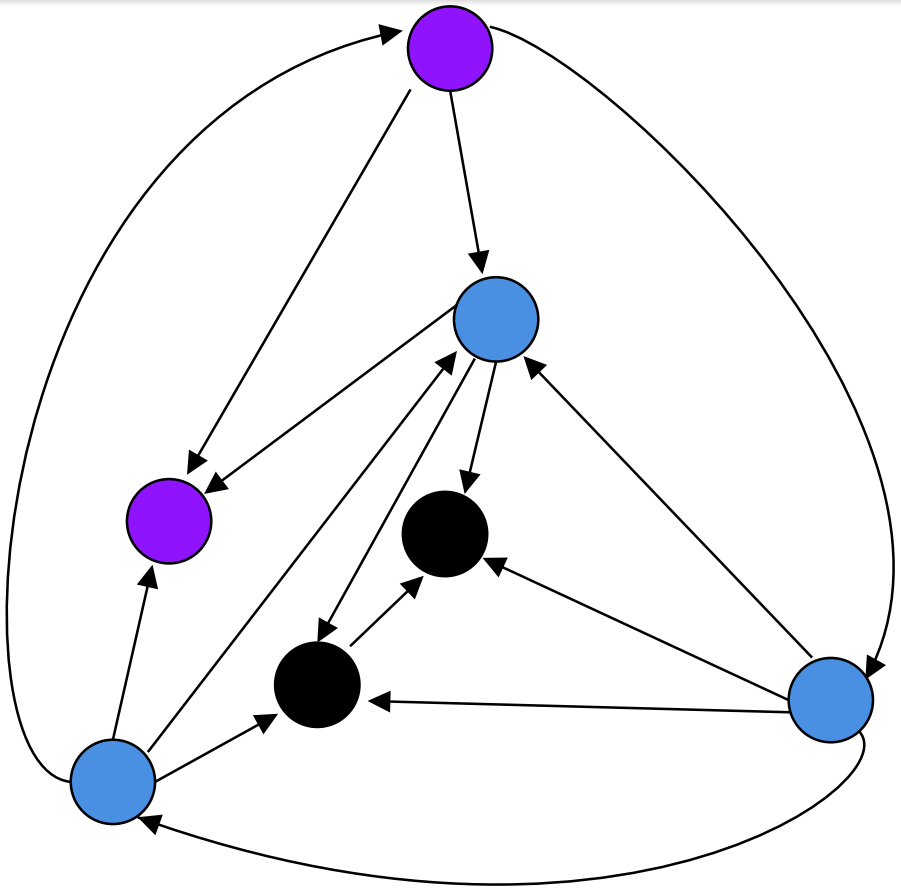
X = vetrices on or outside T

X contains r_1, r_2, r_3

Application - connectivity - positive cocircuit lemma

Lemma

For any 3-oriented maximal planar graph G having its external face oriented as a circuit, there exists a bijection between elementary positive cocircuits of G and undirected triangles of G.



$\leftarrow$

T = non empty triangle

X = vetrices on or outside T

X contains $r_1,\ r_2,\ r_3$

summing indegrees,

$\omega(X)$ is a positive cocircuit

Application - 4-connectivity of maximal planar graphs

Lemma

A 3-oriented maximal planar graph $G = (V, E)$, where $|V| \geq 5$, having its external face oriented as a circuit is 4-connected iff all its positive cocircuits are trivial (defined by a single face).

Application - 4-connectivity of maximal planar graphs

Lemma

A 3-oriented maximal planar graph $G = (V, E)$, where $|V| \geq 5$, having its external face oriented as a circuit is 4-connected iff all its positive cocircuits are trivial (defined by a single face).

A shelling order can be computed in $O(|V|)$, it induces a Schnyder decomposition, which yields 3-orientation.

Application - 4-connectivity of maximal planar graphs

Lemma

A 3-oriented maximal planar graph $G = (V, E)$, where $|V| \geq 5$, having its external face oriented as a circuit is 4-connected iff all its positive cocircuits are trivial (defined by a single face).

A shelling order can be computed in $O(|V|)$, it induces a Schnyder decomposition, which yields 3-orientation.

To test, if it has a nontrivial positive cocircuit, remove r_1, r_2, r_3, compute dual oriented graph and test if there is a cycle.

Maximal bipartite planar analogs

Definition

Cocircuit $\omega(X)$ is semi-elementary iff X is in one face region of $\overline{X}$ and vice versa/

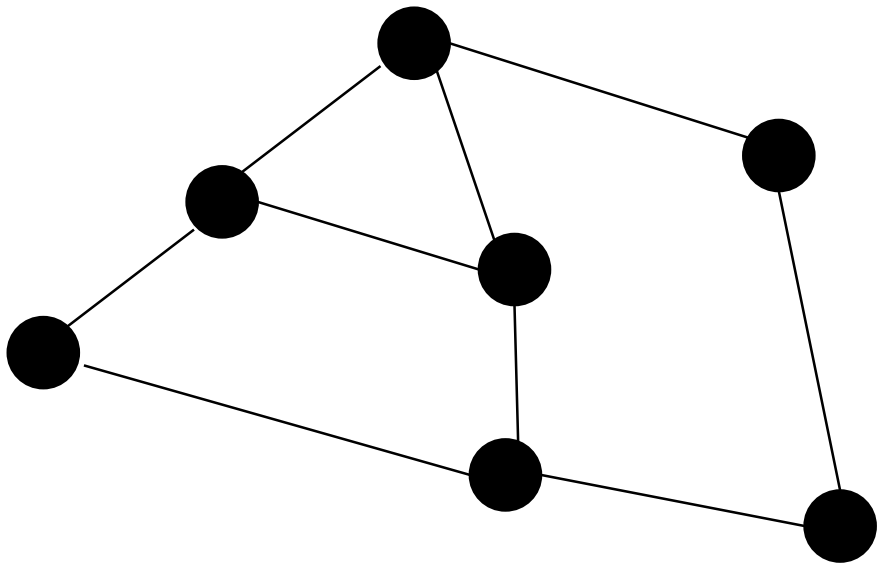
Lemma

A 2-oriented maximal bipartite planar graph having its external face oriented as a circuit. The quadrangles of G without a chord are in bijection with semi-elementary positive cocircuits.

Angle graph

Definition

Angle graph of a 2-connected planar graph G is a incidence graph between V and F of G.

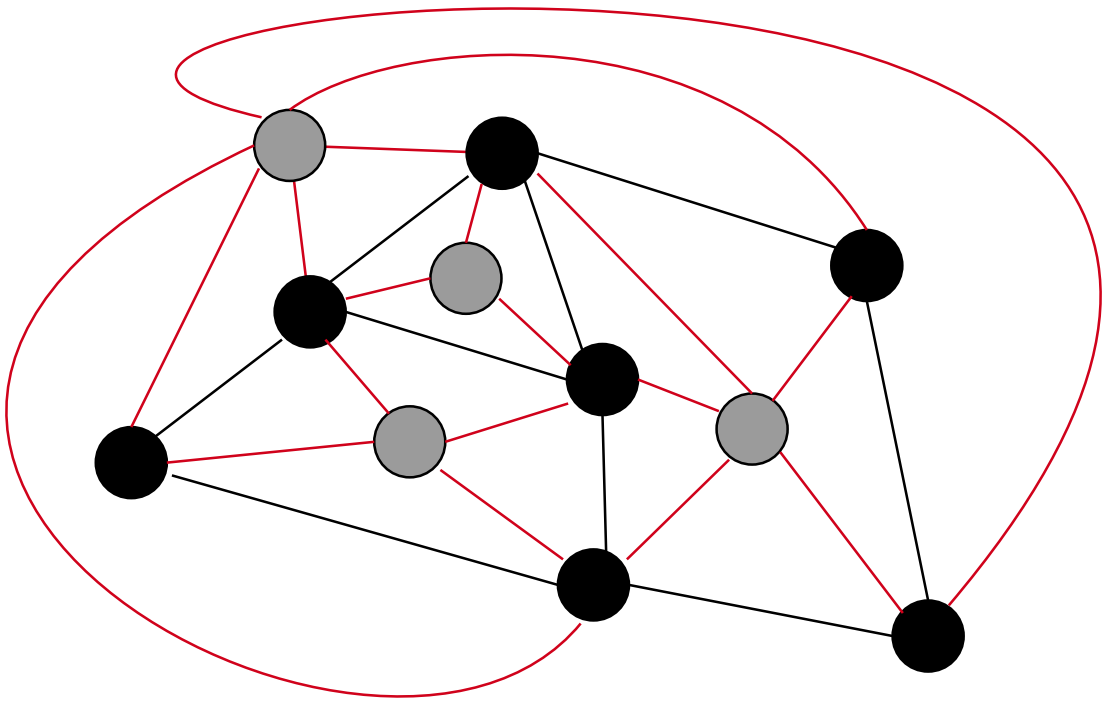


Angle graph

Definition

Angle graph of a 2-connected planar graph G is a incidence graph between V and F of G.

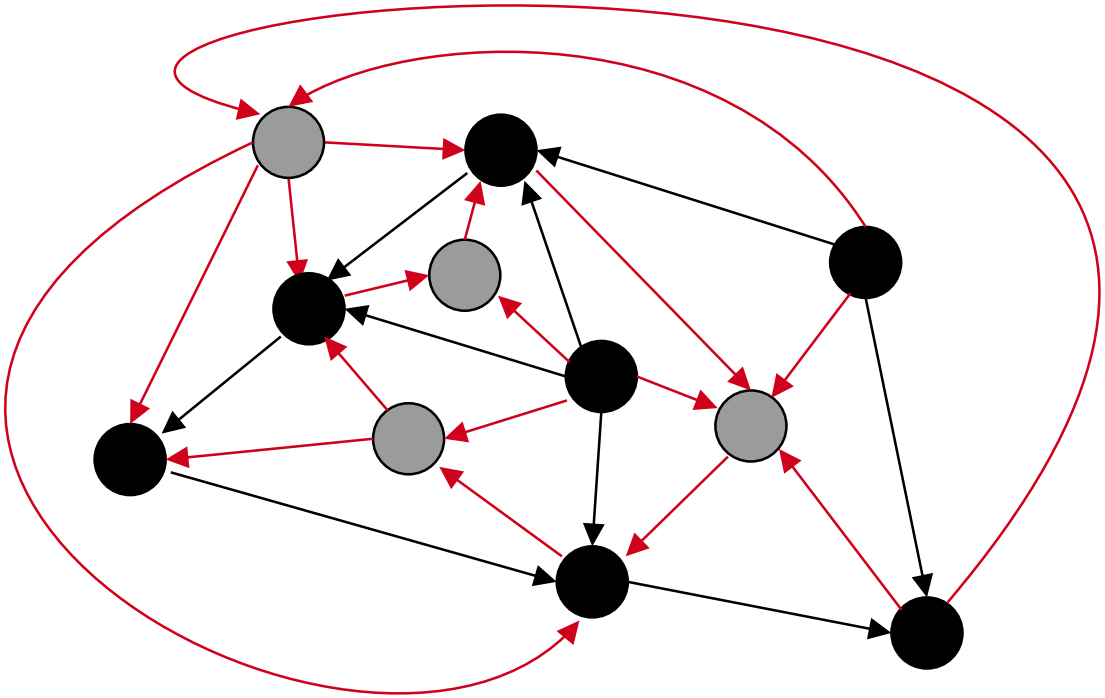
It is always maximal bipartite planar.



Directed Angle graph

Edge in $A(G)$ is pair of two consecutive edges of G from one vertex in clockwise order:

- if one is incoming, and other is outgoing, then edge from face to vertex in $A(G)$
- otherwise in reverse direction



Computing semi-elementary positive cocircuits of $A(G)$

(s, t)-bipolar orientation - acyclic orientation with exactly one source s and sink t, linked by an edge (s, t).

Computing semi-elementary positive cocircuits of $A(G)$

(s, t)-bipolar orientation - acyclic orientation with exactly one source s and sink t, linked by an edge (s, t).

For any orientation of biconnected G, let (r_1, r_3) be any directed edge. That (r_1, r_3) is an bipolar orientation iff the induced orientation in $A(G)$ is a 2-orientation with external edges oriented from r_1 and r_3.

3-connectivity of any planar graph

Lemma

Let G be a 2-connected planar graph and $A(G)$ its angle graph.
If we 2-orient $A(G)$, with external face as a circuit, then G is 3-connected iff $A(G)$ has only one semi-elementary cocircuit – its external face.

3-connectivity of any planar graph

Lemma

Let G be a 2-connected planar graph and $A(G)$ its angle graph.
If we 2-orient $A(G)$, with external face as a circuit, then G is 3-connected iff $A(G)$ has only one semi-elementary cocircuit – its external face.

2-articulated subgraph of G is a proper induced subgraph on at least 3 vertices, that could arise from removing articulation pair in G.

3-connectivity of any planar graph

Lemma

Let G be a 2-connected planar graph and $A(G)$ its angle graph.
If we 2-orient $A(G)$, with external face as a circuit, then G is 3-connected iff $A(G)$ has only one semi-elementary cocircuit – its external face.

2-articulated subgraph of G is a proper induced subgraph on at least 3 vertices, that could arise from removing articulation pair in G.

Semi-elementary positive cocircuits of $A(G) \leftrightarrow$ quadrangles in $A(G) \leftrightarrow$ 2-articulated subgraphs of G.

One more interesting fact

Lemma

For 2-connected planar graph G and 2-orientation of $A(G)$, the possible extensions by edge additions of $A(G)$ to 3-oriented maximal planar graphs are in bijection with the spanning trees of G.