Isomorphic bisections of cubic graphs[1]

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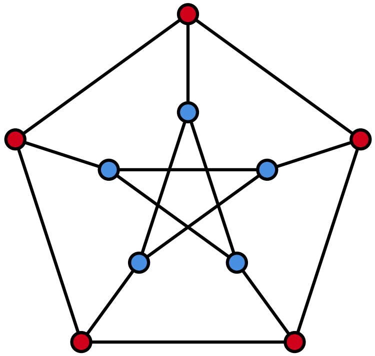
[1] S. Das, A. Pokrovskiy, and B. Sudakov. "Isomorphic bisections of cubic graphs". In: *Journal of Combinatorial Theory, Series B* 151 (2021), pp. 465–481.

Conjecture (Ando)

The vertices of any cubic graph can be 2-colored such that the two color classes induce isomorphic subgraphs.



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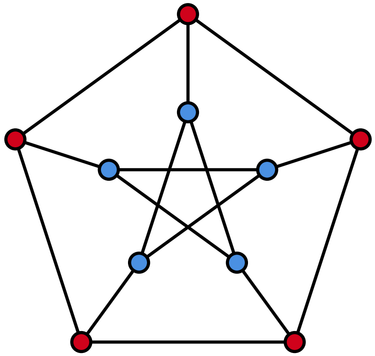

Introduction

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Main Theorem

Every sufficiently large connected cubic graph admits a 2-coloring ϕ whose classes induce isomorphic subgraphs.

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Conclusion

There can be at most finitely many counterexamples to Ando's conjecture.

Notation

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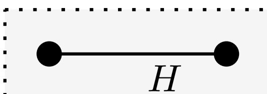
- Let H be a fixed graph, let φ be a red-blue-colouring on vertices of another graph G.
  We define $r_H(G, \varphi)$ as the number of red components of G under φ that are isomorphic to H.



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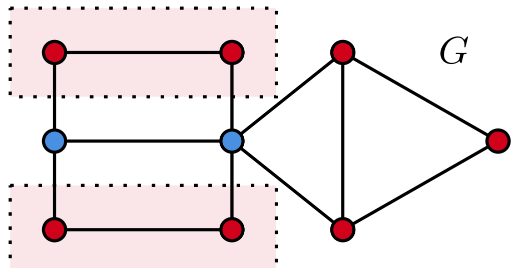

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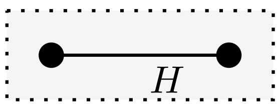

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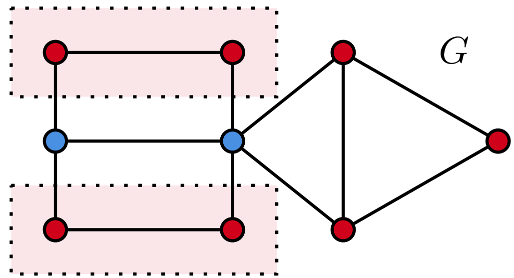
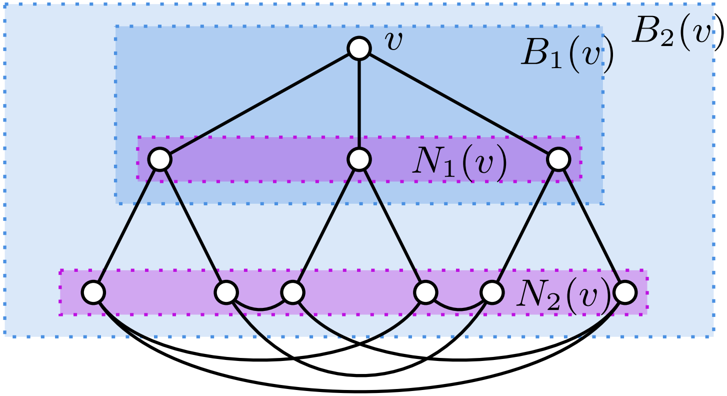


- We define $b_H(G, \varphi)$ similarly for blue components of G.

Notation

- We define $B_d(v)$ as the d-radius ball around vertex v.
- We define $N_d(v)$ as the set of vertices at distance exactly d from v.



- We define P_t as path of length $t - 1$.

P_2

P_3

P_5

Proof of the main theorem road map

The goal is to color the vertices of a large cubic graph so that the color classes induce isomorphic subgraphs. We do this in two stages:

I. We take semi-random vertex coloring and show that this is very close to having desired properties.

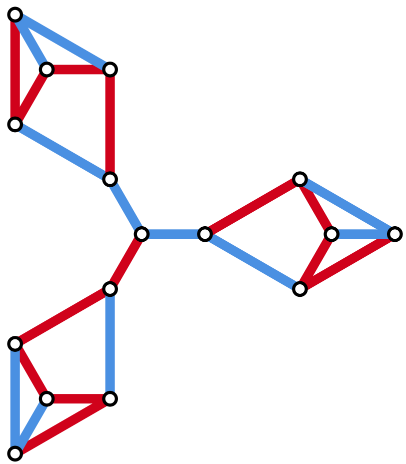
II. We make deterministic local recoloring to balance the two subgraphs and ensure they are truly isomorphic.

Stage I: Tools

Theorem (Thomassen)

The edges of any cubic graph can be two-colored such that each monochromatic component is a path of length at most five.Thomassen[a]

[a] Carsten Thomassen. "Two-Coloring the Edges of a Cubic Graph Such That Each Monochromatic Component Is a Path of Length at Most 5". In: *Journal of Combinatorial Theory, Series B* 75.1 (1999), pp. 100–109.



Stage I: Tools

Theorem (McDiarmid's Inequality)

Let $X = (X_1, X_2, ..., X_n)$ be a family of independent random variables, with X_k taking values in a set A_k for each $k \in [n]$. Suppose further that there is some function f satisfying:

1. $f : A_1 \times A_2 \times ... \times A_n \to \mathbb{R}$
2. there exists $c \leq 0$ such that $|f(x) - f(x')| \leq c$ whenever x, x' differ in a single coordinate.

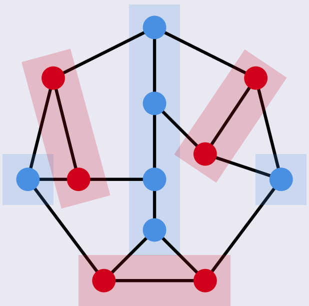
Then, for any $m \leq 0$:

$$P(|f(X) - E[f(X)]| \geq m) \leq 2\exp(\tfrac{-2m^2}{c^2 n})$$

Proposition 2.1

For any $d \in \mathbb{N}$ and sufficiently large cubic graph G, there is a red-blue-colouring φ for which the following holds:

(a) Each monochromatic component is a path of length at most 5.



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(c) There are sequences of vertices $(u_i)_{i \in [s]}$ and $(w_i)_{i \in [s]}$, for some $s \geq 2^{-2d-5}n$, such that all balls $B_d(u_i)$ and $B_d(w_i)$ are pairwise disjoint, and for each $i \in [s]$, induce isomorphic subgraphs with opposite colourings.

Proof Idea.
We will define a random coloring φ' and show that it satisfies a, b and c.
We then will make small adjustments to φ' and make it satisfy (d).

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(d) φ is a bisection, that is there is an equal number of red and blue vertices.

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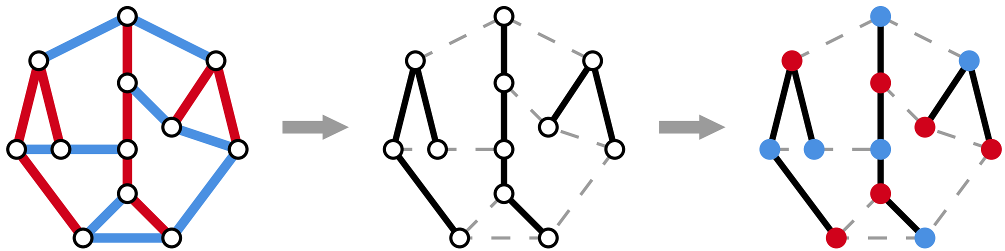
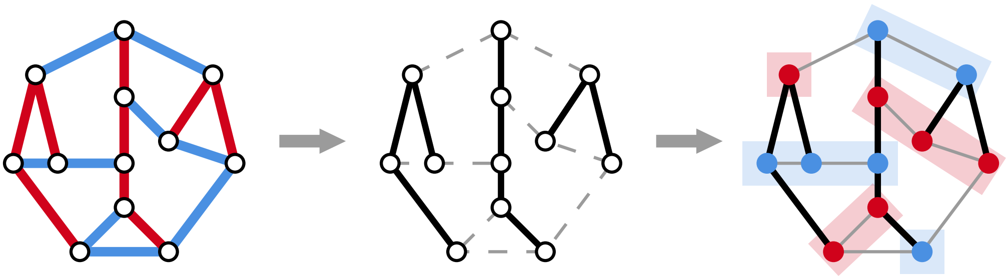


- We apply Thomassen's theorem to partition edges G into two spanning linear forests F_1, F_2
(every path has a length of less than 6).
- We take φ' to be a uniform random proper 2-coloring of F_1.

Each path in F_1 has two possible colorings and each path can be colored independently. Thus probability space is a product space.

Coloring φ' satisfies (a):

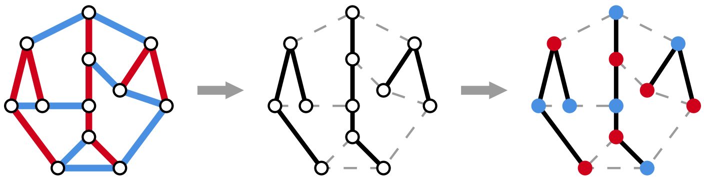
Each monochromatic component is a path of length at most 5



Because φ' is a proper coloring of F_1 each monochromatic component of G has to be a subgraph of F_2. It follows immediately that such a component is a path of length at most 5.

Coloring φ' satisfies (b):

For $2 \leq t \leq 6$, $|r_{P_t}(G,\varphi) - b_{P_t}(G,\varphi)| \leq 3\sqrt{n \log n}$



We use McDiarmid's inequality.

$$P(|f(X) - E[f(X)]| \leq m) \leq 2\exp(\tfrac{-2m^2}{c^2 n})$$

To do so we must understand how changing the coloring of a path in F_1 affects $r_{P_t}(G,\varphi')$.

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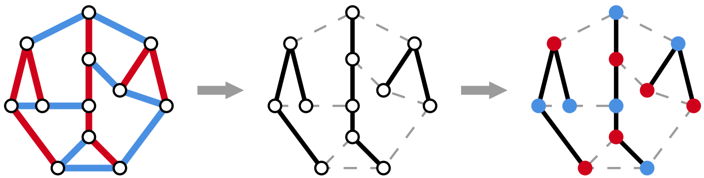

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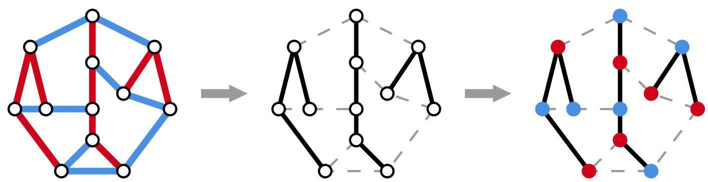

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Because every path in F_1 has at most 6 vertices then reversing its coloring can affect at most <u>12</u> monochromatic copies of P_t.

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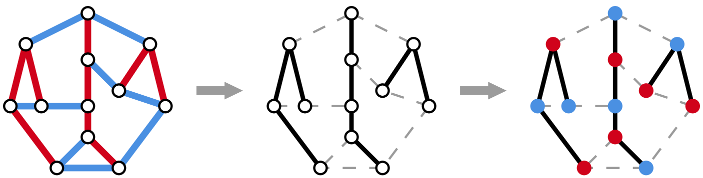

After applying McDiarmid's inequality (for right m) we get:

$$P(|r_{P_t}(G,\varphi') - E[r_{P_t}(G,\varphi')]| \geq \sqrt{n \log n}) \leq 2n^{-1/72} = o(1)$$

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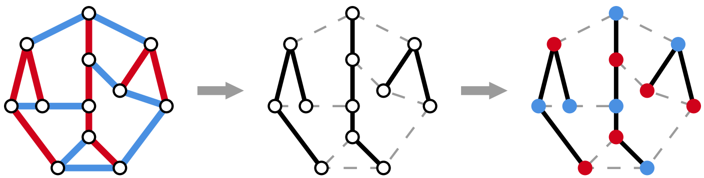

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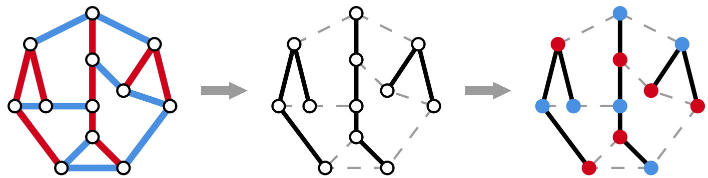

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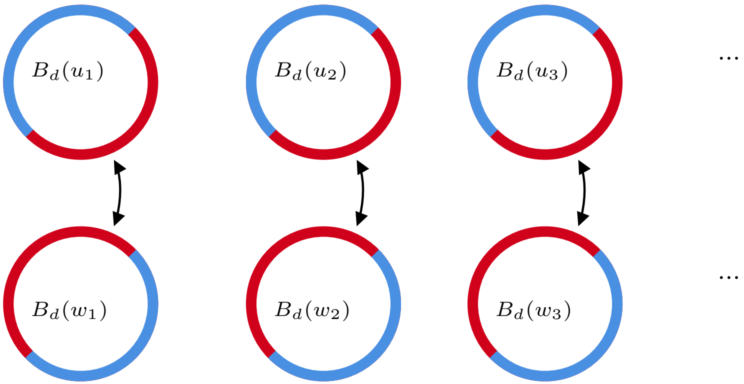
Hence we get that:

$$P(|r_{P_t}(G,\varphi') - b_{P_t}(G,\varphi')| \ge 2\sqrt{n \log n}) = o(1)$$

Coloring φ' satisfies (c):

There are sequences of vertices $(u_i)_{i\in[s]}$ and $(w_i)_{i\in[s]}$, for some $s \geq 2^{-2d-5}n$, such that all balls $B_d(u_i)$ and $B_d(w_i)$ are pairwise disjoint, and for each $i \in [s]$, induce isomorphic subgraphs with opposite colourings.



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Proof sketch:

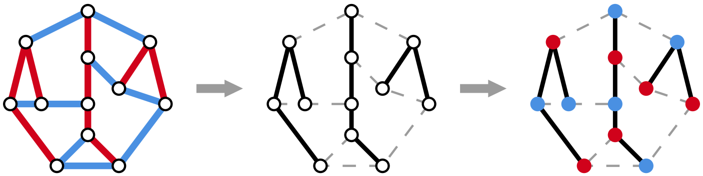
- For a large graph we can pick a set of independent balls with size $s' \geq 2^{-2d-1}n$.
- There are a finite number of possible classes of colored balls.
- We can show that with $1 - o(1)$ probability for every such class C:

  $$|Y - \overline{Y}| \leq 2\sqrt{n \log n}$$

  Where Y is the number of balls belonging to class C, and $\overline{Y}$ is the number of balls belonging to the class with opposite coloring.
- We can therefore match 1/4 balls into isomorphic pairs with opposite colorings.

Coloring φ' almost satisfies (d):

With probability $1 - o(1)$ the difference between number of red and blue vertices is at most $\frac{1}{20}\sqrt{n \log n}$.



Proof sketch:

- Only even-length paths in F_1 contribute to the difference, and they contribute by exactly 1.
- The expected difference is 0.
- We can apply McDiarmid's inequality to get the bound.

For large n with high probability φ' satisfies a, b, c and almost d.

We can create φ from φ' that will satisfy a, b, c and d.

Proof sketch:

- Let Δ be the difference between red and blue vertices.
- We can reverse the coloring of Δ even-length paths in F_1.

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- Let Δ be the difference between red and blue vertices.
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(b) For each $t \in \{2, ..., 6\}$ we will change $|r_{P_t}(G, \varphi) - b_{P_t}(G, \varphi)|$ by at most $\sqrt{n \log n}$. So (b) will still be satisfied.

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(c) Similarly φ will affect some balls of φ' but at least $2^{-2d-5}n$ pairs will prevail. Thus (c) holds as well.

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This completes the proof of proposition 2.1 ■

Stage II: Correcting the bisection

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Definition: P_t-reducer

Given $t \geq 3$, an induced subgraph $R \subseteq G$ is a P_t-reducer if there are two vertex colorings ψ_1, ψ_2 of $B_2(R) = R \cup N(R) \cup N^2(R)$ such that.

(i) ψ_1, ψ_2 have the same number of red vertices (and therefore blue as well).

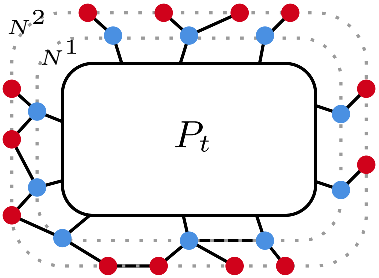
(ii) In both ψ_1, ψ_2 $N(R)$ are colored blue and $N^2(R)$ are colored red.

(iii) $r_H(B_2(R), \psi_1) = r_H(B_2(R), \psi_2)$ and $b_H(B_2(R), \psi_1) = b_H(B_2(R), \psi_2)$, unless $H = P_\ell$ for some $2 \leq \ell \leq t$

(iv) $r_{P_t}(B_2(R), \psi_2) = r_{P_t}(B_2(R), \psi_1) - 1$ and $b_{P_t}(B_2(R), \psi_2) = b_{P_t}(B_2(R), \psi_1)$

Intuition. For a P_t-reducer and colorings ψ_1, ψ_2:

- The only monochromatic components that can appear/disappear are paths of length $\leq t$.
- The number of monochromatic paths o length t changes by exactly 1.



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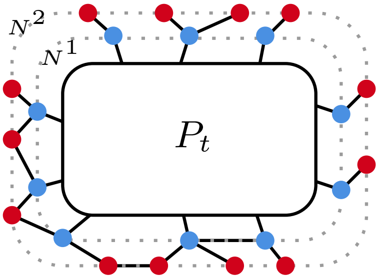

Theorem 2.4

Let G be a cubic graph on more than $3 \cdot 2^{50}$ vertices, and let $v \in V(G)$. Then for every $3 \leq t \leq 6$, there is a P_t-reducer in $B_{50}(v)$.

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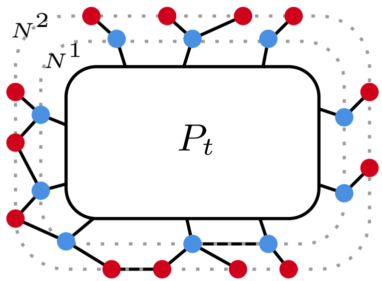

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Proof omitted.

Proof of the main theorem

Main Theorem

Every sufficiently large connected cubic graph admits a 2-coloring φ whose classes induce isomorphic subgraphs.

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Set $d = 57$ and use it in proposition 2.1. We get a 2-coloring φ_0, such that:

- $r_H(G, \varphi_0) = b_H(G, \varphi_0)$ except $H = P_t, 2 \leq t \leq 6$.
- $|r_{P_t}(G, \varphi_0) - b_{P_t}(G, \varphi_0)| \leq 3\sqrt{n \log n}$ for $2 \leq t \leq 6$

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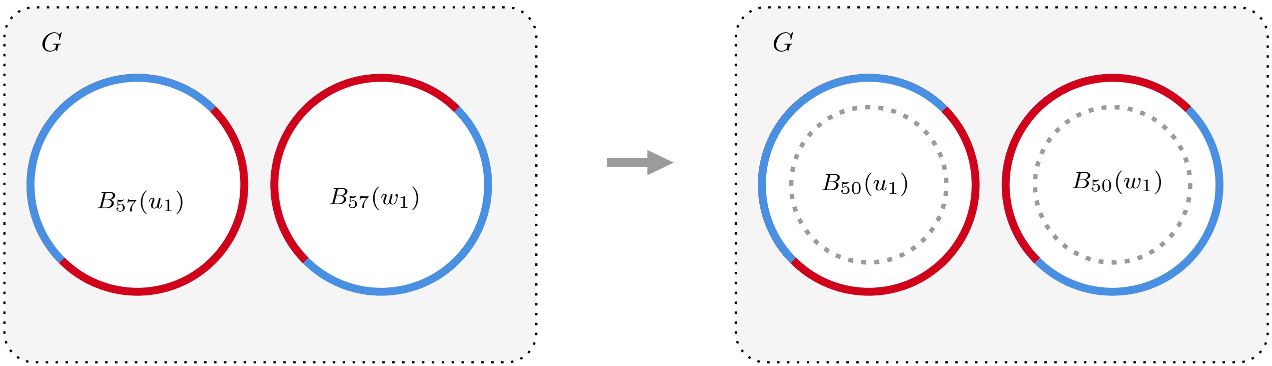

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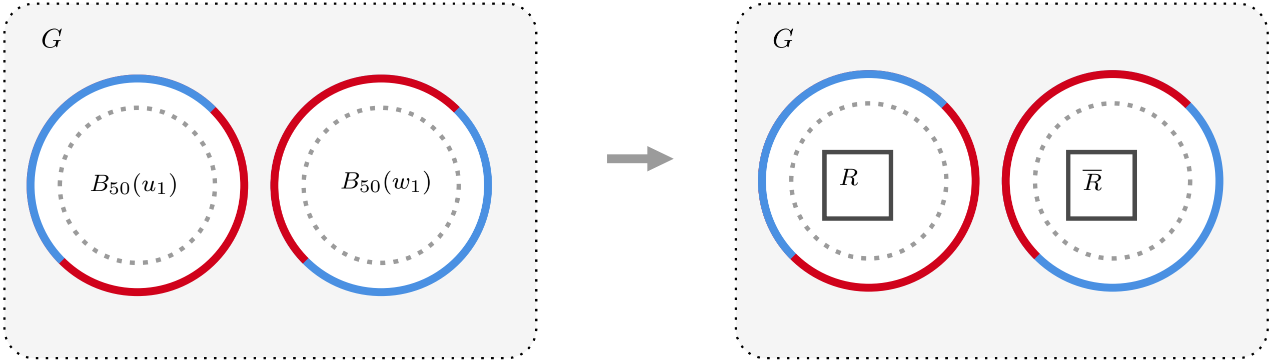
4. By proposition 2.4 we can find a P_6-reducer R in $G[B_{50}(u_1)]$.
5. Therefore w get $\bar{R}$ corresponding oppositely-colored copy of R, such that $\bar{R} \subseteq G[B_{50}(w_1)]$.



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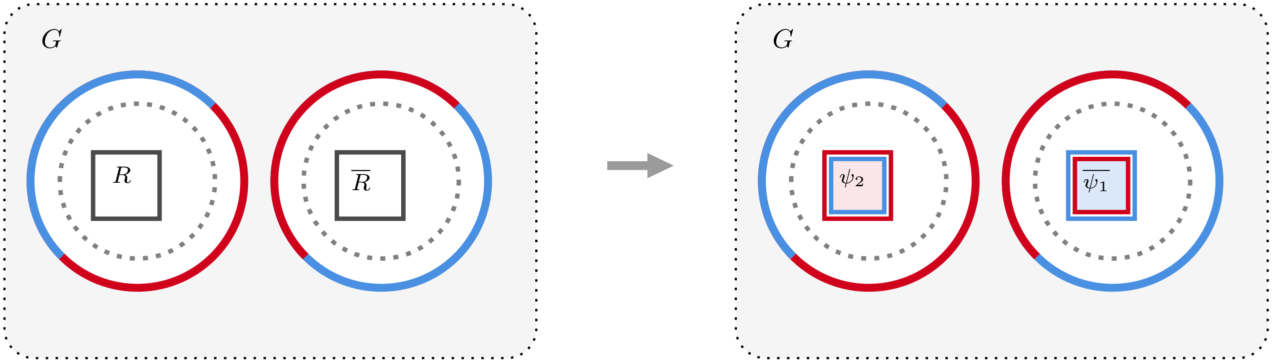
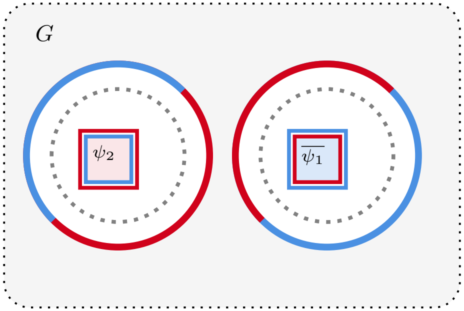
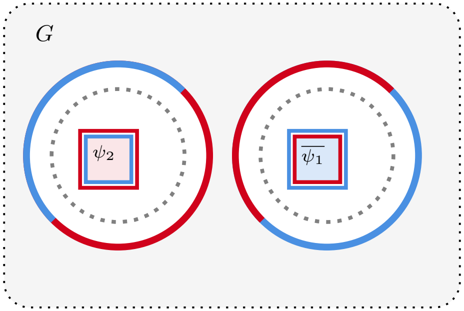
6. Let ψ_1, ψ_2 be the colorings for R that show it's a P_6-reducer.
7. And let $\bar{\psi_1}, \bar{\psi_2}$ be the opposite colorings.
8. Color $B_2(R)$ with ψ_2 and $B_2(\bar{R})$ with $\bar{\psi_1}$, lets call it φ_1.





Observe that ψ_1 is still bisection:

- Colorings ψ_1, ψ_2 have the same number of red vertices.

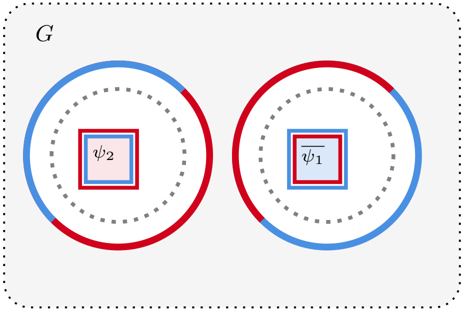


Observe that ψ_1 is still bisection:

- Colorings ψ_1, ψ_2 have the same number of red vertices.

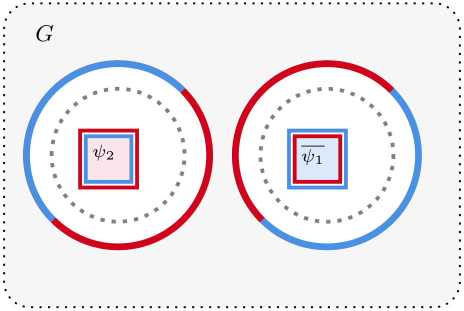
Claim

For all H, $r_H(G, \psi_1) = b_H(G, \psi_1)$, except P_t, $2 \le t \le 6$. And the difference between the number of red and blue P_6 is reduced by 1.



Claim proof sketch.

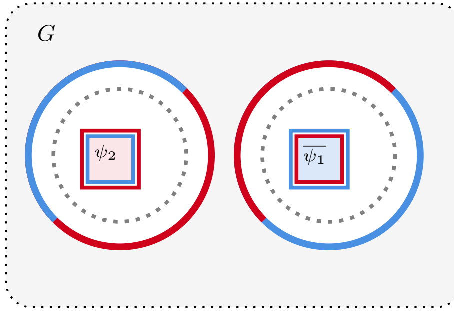
- Monochromatic components could only change in $B_{57}(u_1)$ and $B_{57}(w_1)$. (We made changes in $B_{50}(u_1)$ and $B_{50}(w_1)$)



Claim proof sketch.

- Monochromatic components not fully contained in $B_2(R)$ ($B_2(\bar{R})$) are ok.

  Such components can only contain vertices from $B_2(R)$ but from symmetry, we get blue isomorphic components in the other ball.



Claim proof sketch.

- This leaves monochromatic components fully contained in $B_2(R), B_2(\bar{R})$.

  We use properties of P_6-reducer.

  - For every H except P_t where $2 \leq t \leq 6$:

    $$r_H(B_2(R), \psi_2) = r_H(B_2(R), \psi_1) = b_H(B_2(\bar{R}), \bar{\psi_1})$$

    Similarly:

    $$b_H(B_2(R), \psi_2) = r_H(B_2(\bar{R}), \bar{\psi_1})$$

  - For P_6 we get:

    $$r_{P_6}(B_2(R), \psi_2) = r_{P_6}(B_2(R), \psi_1) - 1 = b_{P_6}(B_2(\bar{R}), \psi_1) - 1$$

    While:

    $$b_{P_6}(B_2(R), \psi_2) = r_{P_6}(B_2(\bar{R}), \psi_1)$$

We repeat this process $r_{P_6}(G, \varphi_0) - b_{P_6}(G, \varphi_0)$ times, every time taking new pair of balls.

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Every such step could only create a constant number of P_5-components. Thus $|r_{P_5}(G, \varphi_k) - b_{P_5}(G, \varphi_k)| = O(\sqrt{n \log n})$. We can continue the process for P_5, P_4, P_3.

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Observe that we will need at most $O(\sqrt{n \log n})$ steps and we have $\Omega(n)$ pairs of balls. Thus for large graphs, we can finish this process.

After this procedure we get that $r_H(G, \varphi) = b_H(G, \varphi)$ for every $H \neq P_2$.

After this procedure we get that $r_H(G, \varphi) = b_H(G, \varphi)$ for every $H \neq P_2$.

Because G is cubic and φ is a bisection, the number of monochromatic red edges is equal to blue.

This yields that $r_{P_2}(G, \varphi) = b_{P_2}(G, \varphi)$, which completes the proof. ■

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