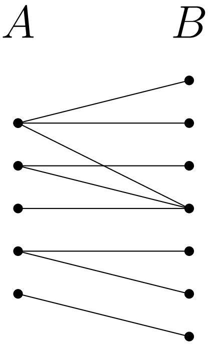
A generalization of König's theorem

L. Lovász

January 5, 2023

Terminology

We will fix sets A and B and only consider bipartite graphs joining them.



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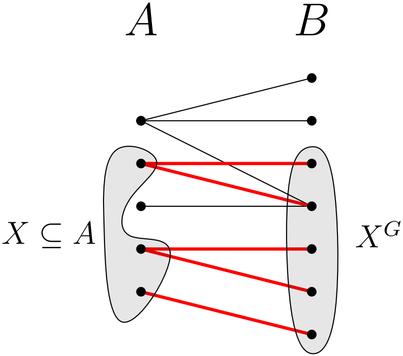
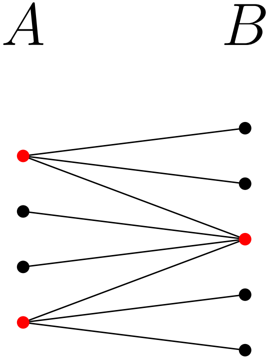
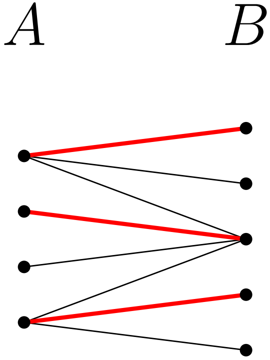


König's theorem

min vertex cover $=$ max matching

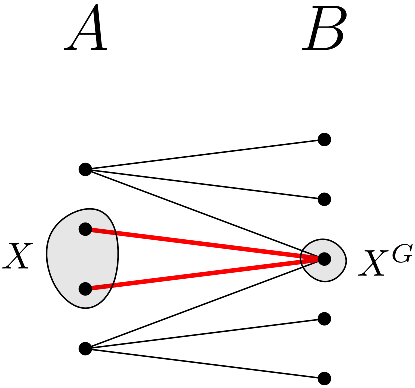




Ore's equivalent theorem

δ – max matching cardinality

$$\delta = |A| - \max_{X \subseteq A}\{|X| - |X^G|\}$$



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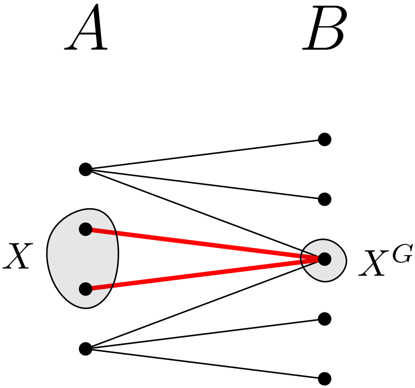

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Hall: $\delta \leftarrow 0$

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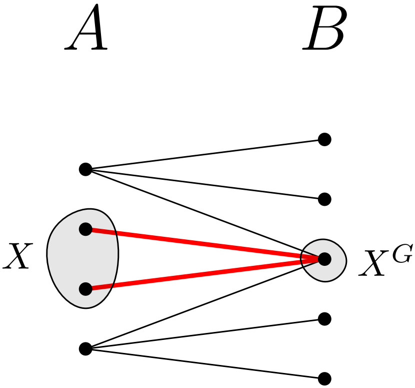

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For $f(X) = |X|$, K^f is the set of graphs admitting an A-perfect matching.

Critical graphs

Definition

Given a class of graphs K^f, a graph $G = (\mathbf{V}, E)$ is critical in K^f, iff for every graph $G' = (\mathbf{V}, E')$ where $E' \subsetneq E$, $G' \notin K^f$.

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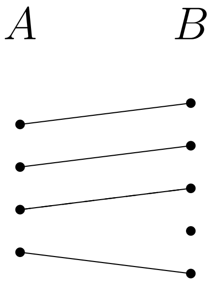

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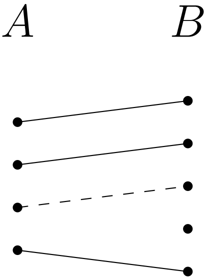

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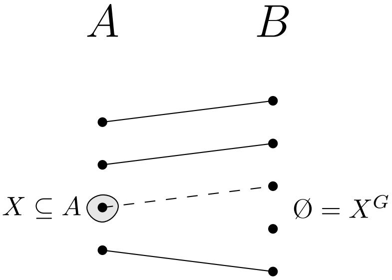

Which graphs are critical?

Theorem 1

Assuming

$$\begin{aligned} f(X) + f(Y) &\leq f(X \cup Y) + f(X \cap Y), && \text{if } X \cap Y = \varnothing \\ f(X) + f(Y) &\geq f(X \cup Y), && \text{if } X \cap Y \neq \varnothing \end{aligned}$$

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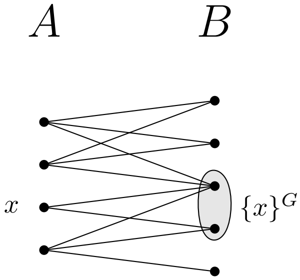

To be in K^f: $\forall_{X \subseteq A} f(X) \leq |X^G|$

To be critical in K^f: $\forall_{x \in A} f(\{x\}) = |\{x\}^G|$

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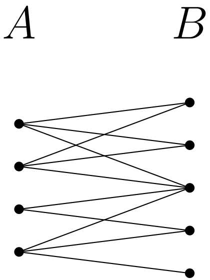

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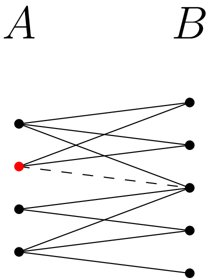

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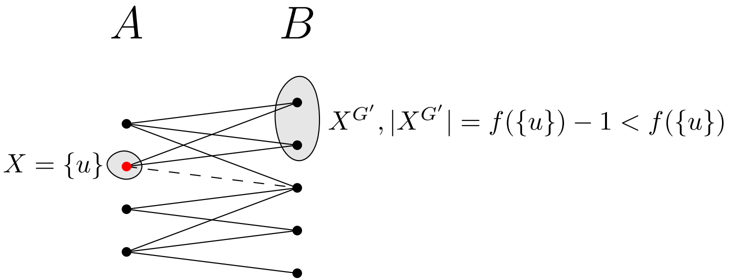

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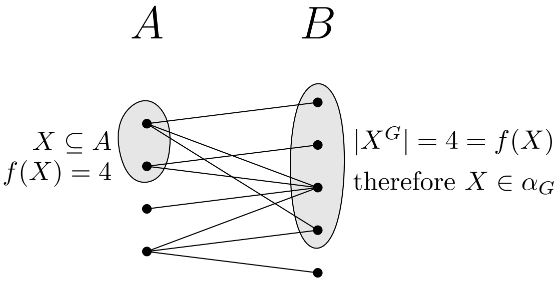

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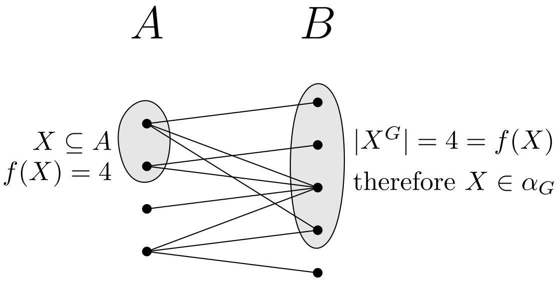

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If $X, Y \in \alpha_G$, then $X \cap Y \in \alpha_G$

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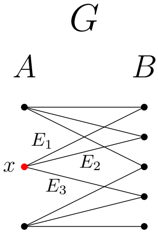

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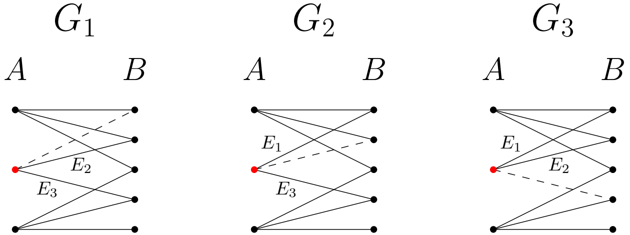

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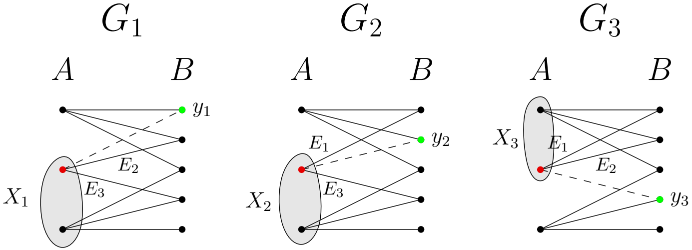

X_i – violator, $\quad x \in X_i, \quad y_i \notin X_i^{G_i}, \quad X_i \in \alpha_G$

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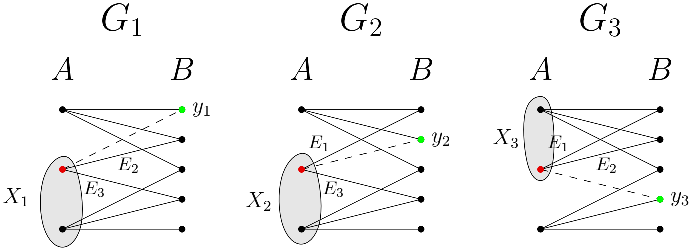

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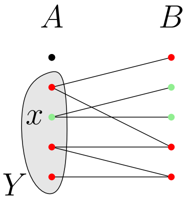

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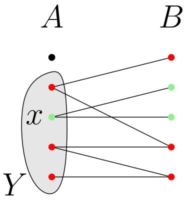

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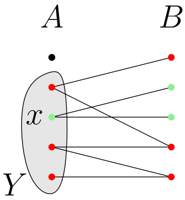

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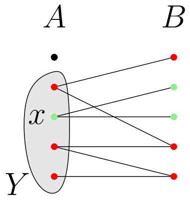

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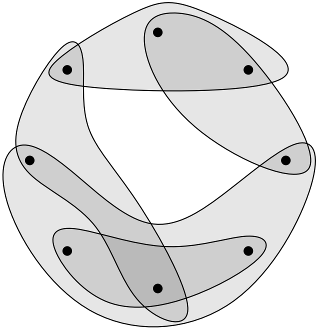
Therefore $\varphi = f(\{x\})$, Q.E.D.

Hypergraphs

Definition

A hypergraph is a set of sets $\mathfrak{H}$.



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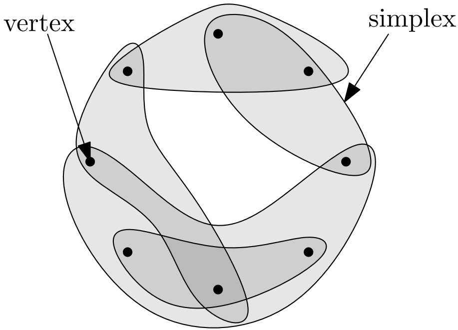

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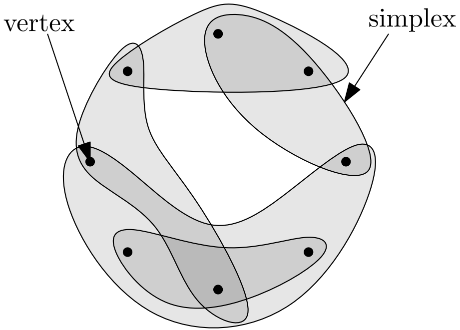

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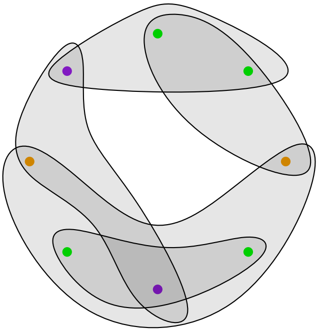
We denote the set of vertices of $\mathfrak{H}$ as $P(\mathfrak{H})$.

Correct coloring

Definition

A correct α-coloring of hypergraph $\mathfrak{H}$ is an assignment of colors $[1..\alpha]$ to vertices such that no simplex is monochromatic.

The chromatic number of hypergraph $\mathfrak{H}$ is the smallest α for which an α-coloring.



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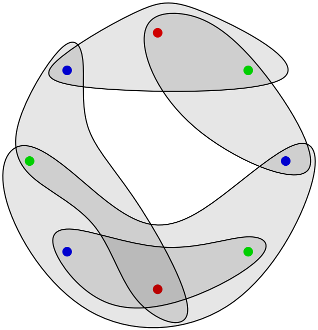

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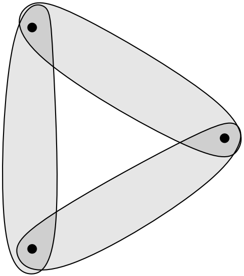

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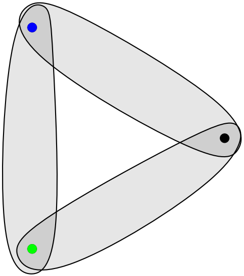

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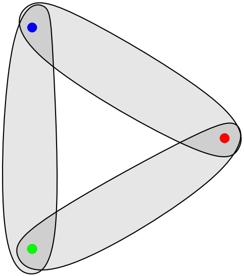

Hypergraph properties

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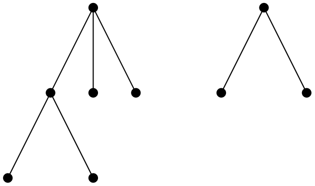

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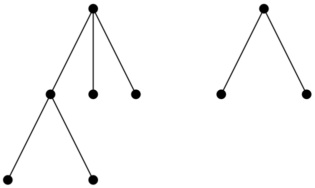

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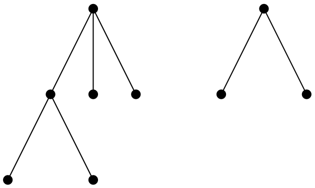

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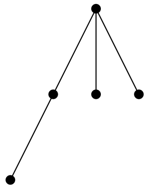

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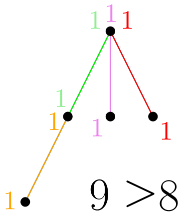

$$9 \geq 8 + 1$$

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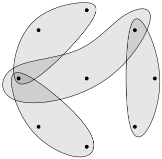
(iii) for every $\mathfrak{H}' \subseteq \mathfrak{H}, \mathfrak{H}' \neq \varnothing$, $|\mathfrak{H}'| + |P(\mathfrak{H}')| \geq \sum_{E \in \mathfrak{H}'} |E| + 1$

(ii) and (iii) are equivalent even when considering any hypergraphs.



Theorem 2

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If a hypergraph $\mathfrak{H}$, for every $\mathfrak{H}' \subseteq \mathfrak{H}, \mathfrak{H}' \neq \varnothing$ has the property:

$$|P(\mathfrak{H}')| \geq (\beta - 1)|\mathfrak{H}'| + 1$$

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This is (iii) from the slide before.
(iii) implies (ii) - the statement that allows induction.

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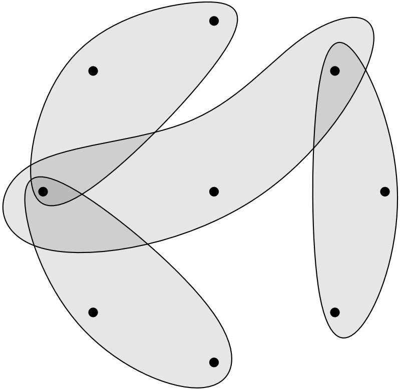

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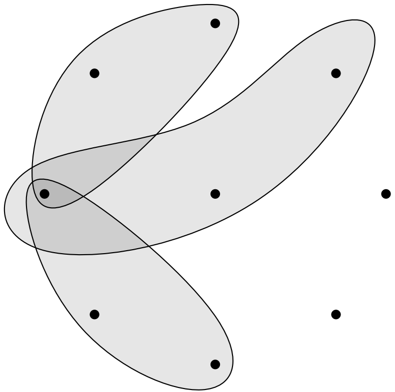

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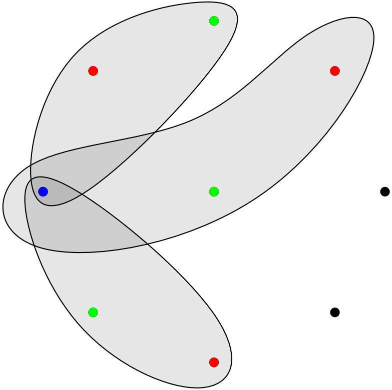

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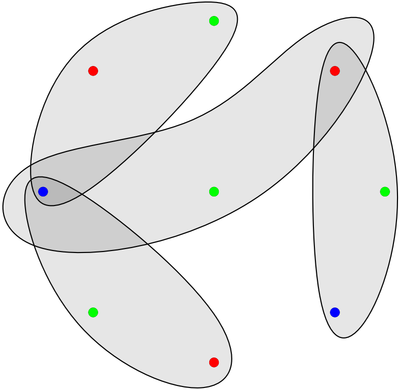
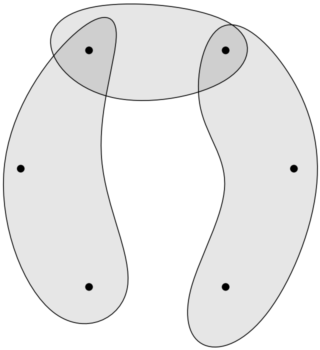


Case II



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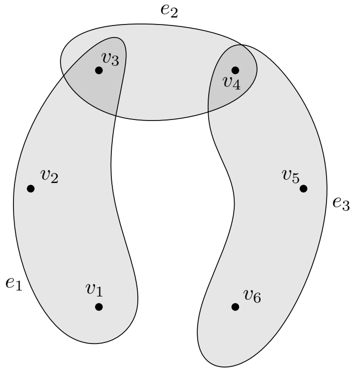

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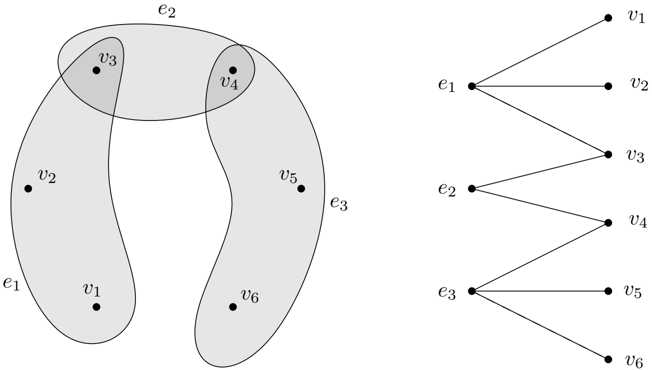

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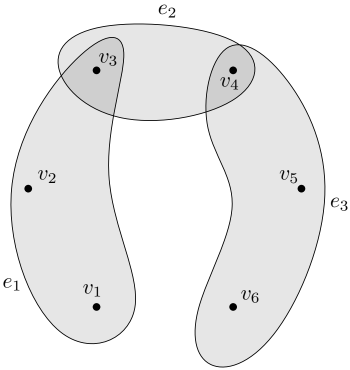

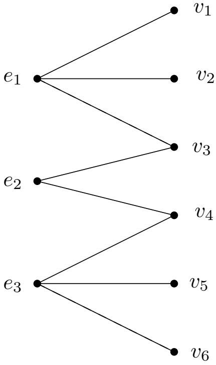

$$f(X) = \begin{cases} (\beta - 1)|X| + 1 & \text{if } X \neq \varnothing \\ 0 & \text{else} \end{cases}$$

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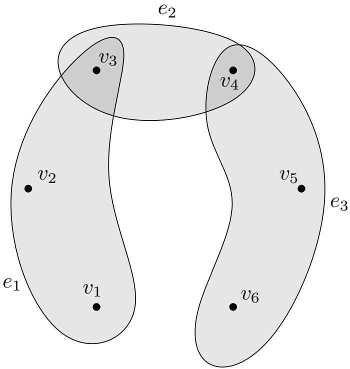

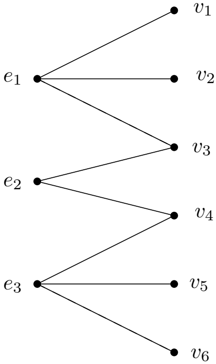

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Critical graph will have $\deg(x) = f(\{x\})$.

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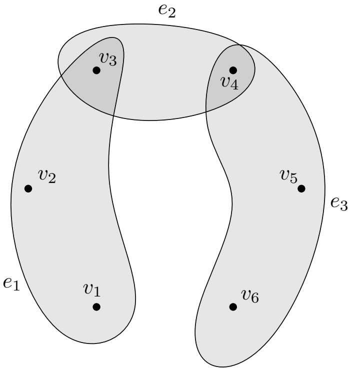

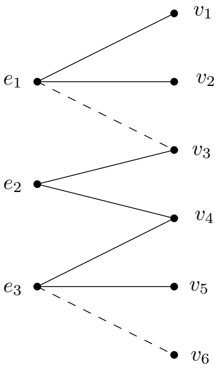

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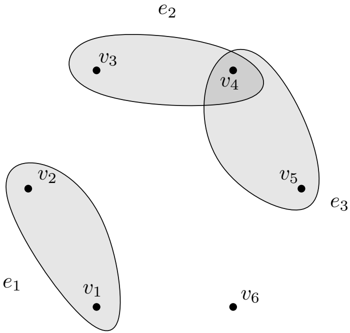

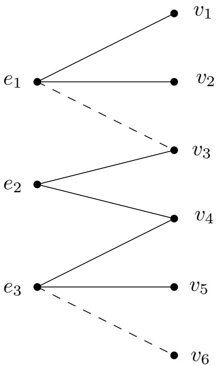

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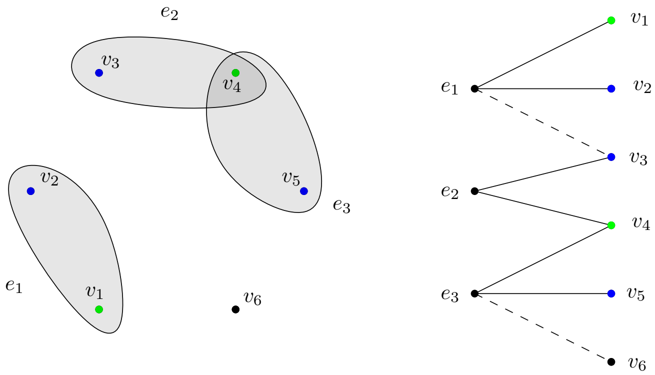

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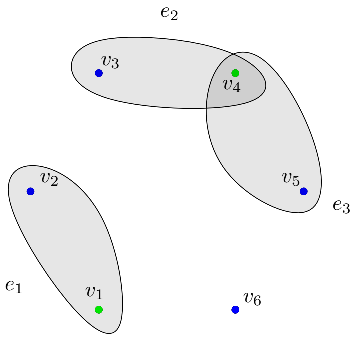

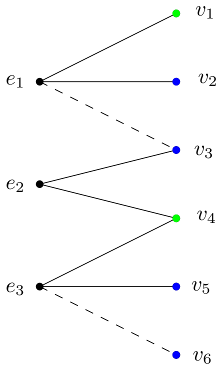

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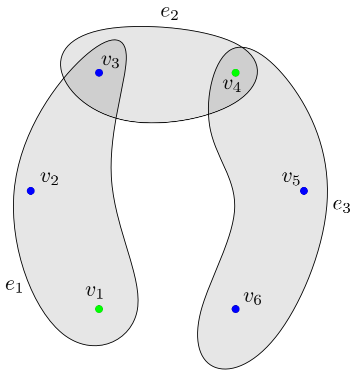

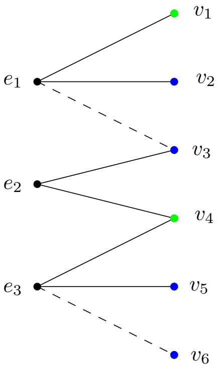

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Critical graph will have $\deg(x) = f(\{x\})$.

Application

Conjecture by Erdos

If a hypergraph $\mathfrak{H}$ fulfills:

for every $\mathfrak{H}' \subseteq \mathfrak{H}, \mathfrak{H}' \neq \varnothing$, $P(\mathfrak{H}') \geq |\mathfrak{H}'| + 1$

Then it has chromatic number ≤ 2.

Colorings

Lemma

A hypergraph $\mathfrak{H}$ has chromatic number ≤ 2 if and only if it has strict chromatic number ≥ 2.

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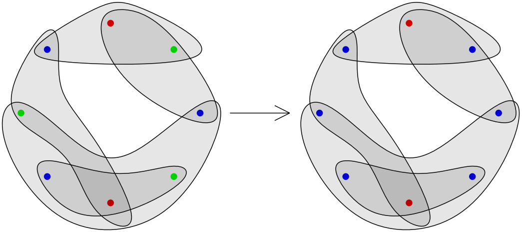

$\Leftarrow$

A strict α-coloring ($\alpha \geq 2$) can be reduced to 2-coloring by mapping $[3..\alpha]$ to 1.

Colorings

Lemma

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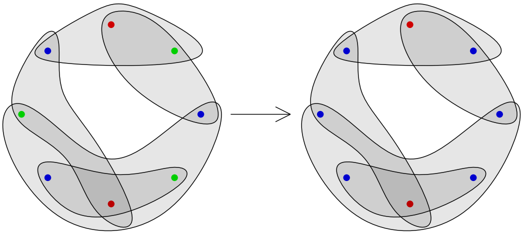

$\Leftarrow$

A strict α-coloring ($\alpha \geq 2$) can be reduced to 2-coloring by mapping $[3..\alpha]$ to 1.

$\Rightarrow$

A 2-coloring is strict by default.

Conjecture proof

$$P(\mathfrak{H}') \geq |\mathfrak{H}'| + 1$$

$$P(\mathfrak{H}') \geq (2-1)|\mathfrak{H}'| + 1$$

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Due to theorem 2, this graph has strict chromatic number ≥ 2, and due to the just proven lemma, it also has chromatic number ≤ 2. Q.E.D.

The end

Thanks for attention!