On induced subgraphs with all degrees odd

by Alexander Scott

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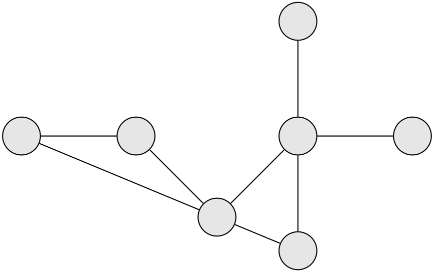

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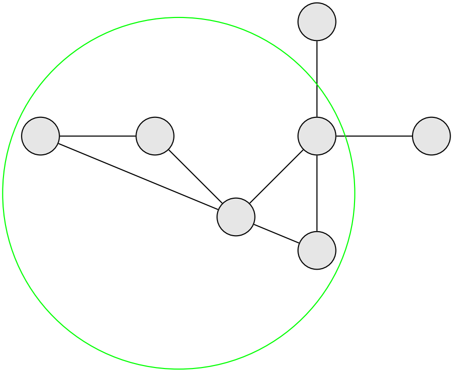

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- $f(n) \geq cn$ - A. Ferber, M. Krivelevich (2022)

Good Partitions

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Given a graph G, we say that a partition $V_1, \ldots, V_k$ of $V(G)$ is a good partition of G if $G[V_i]$ has all degrees odd for $i \in [k]$

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Let G be a graph. Then G has a good partition if and only if every component of G has even order.

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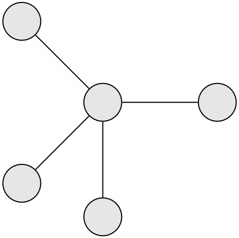

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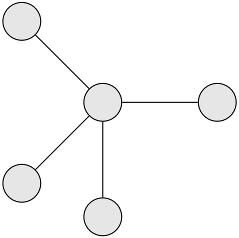

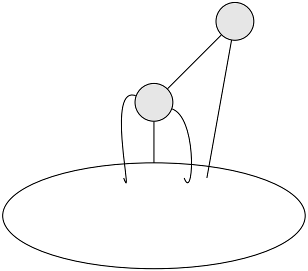

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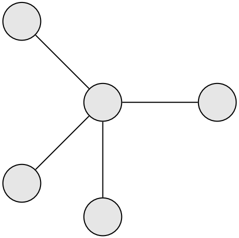

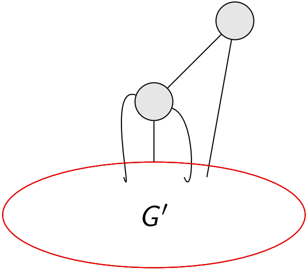

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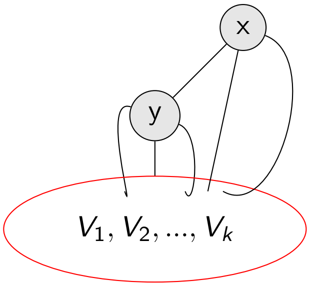

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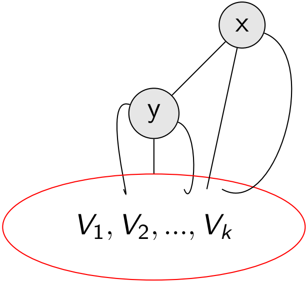

If at least one of V_i has even order, we can take G, and $G \setminus V_i$, if not, then all of V_i have odd order, so there is an even number of V_i

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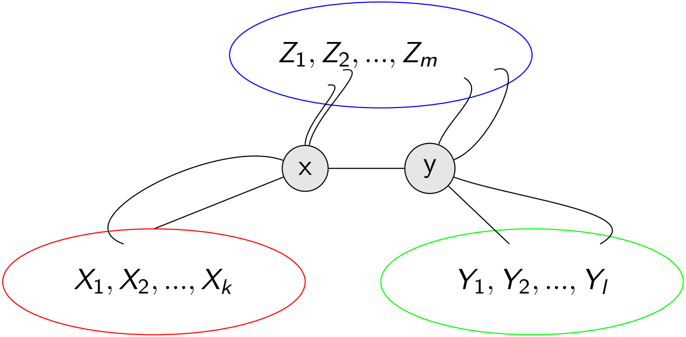

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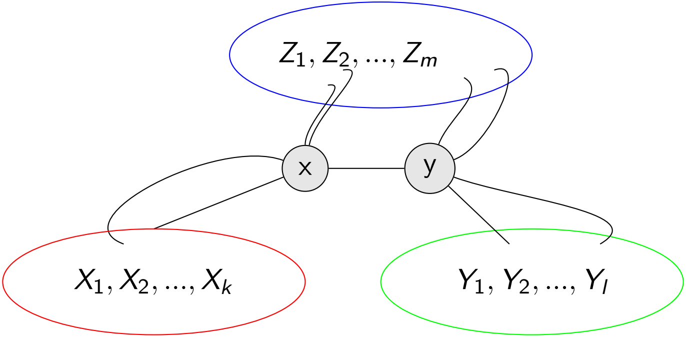

- $X = \{x\} \cup \bigcup_{i=1}^{k} V(X_i) \cup \bigcup_{i=2}^{m} V(Z_i)$
- $Y = \{y\} \cup \bigcup_{i=1}^{l} V(Y_i)$
- One of $|X|$ or $|Y|$ is odd, so we add Z_1 to that set.

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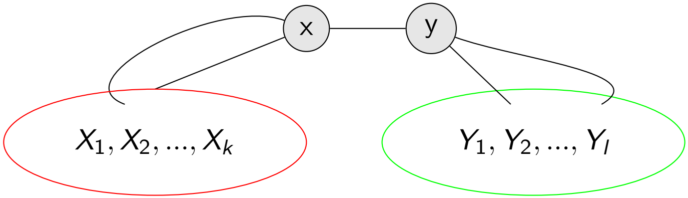

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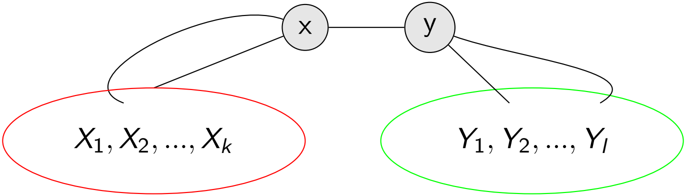

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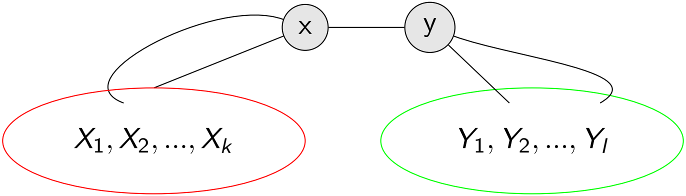

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- $X = \{x, y\} \cup \bigcup_{i=1}^{k} V(X_i)$
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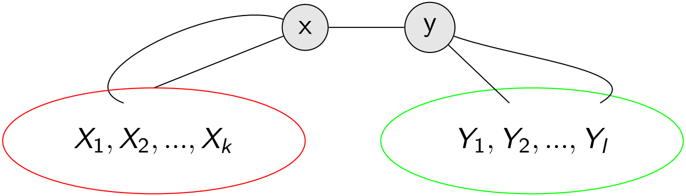

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- $X = \{x, y\} \cup \bigcup_{i=1}^{k} V(X_i)$
- $Y = \{x, y\} \cup \bigcup_{i=1}^{l} V(Y_i)$
- Take good partition of $G[X]$ - W_i and $G[Y]$ - V_i. If $x \in W_1$, and $y \in V_i$, then $x \in V_1$ and $y \in W_1$, so take union of W_1 and V_1

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Lemma

A graph G is a basic odd graph if and only if it is a tree with all degrees odd.

Corollary

Let G be a graph, and suppose that every component of G has even order. Then G has a vertex partition such that every vertex class induces a tree with all degrees odd.

Good partition bounds

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$p(G) = \min\ \{k : G \text{ has a good partition with k sets }\}$

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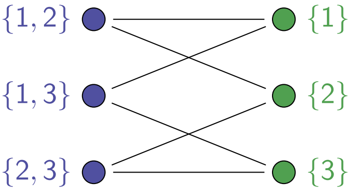
Lemma

$p(n) \geq (1 + o(1))\sqrt{2n}.$

Lemma proof

Let B_n be a biparte graph, with vertex classes $V_0 = [n]^{(1)}$ and $V_1 = [n]^{(2)}$, and edges from $\{i\}$ to $\{j, k\}$ iff $i \in \{j, k\}$.

B_3 :



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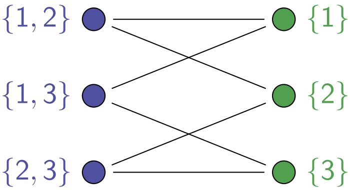

$|B_n| = (1 + \frac{1}{n})\frac{n^2}{2}$., so when the order of B_n is even, we can use the first theorem, and find a good partition. Then all singletons have to be in seperate components, because if some $\{i\}$, $\{j\}$ were in one group, then $\{i,j\}$ can't be partitioned. Thus $p(n) \geq (1 + o(1))\sqrt{2n}$. after some calculations.

Good partition bounds

Theorem

$p(n) \leq cn(log\,logn)^{-\frac{1}{2}}$.

It can be proven that for sufficiently large n, we can find a set W such that $|W| \geq (log\,logn)^{\frac{1}{2}}$, W has all degrees odd, and a good partition. Removing such sets from G leads to upper bound.

Vertex coverings

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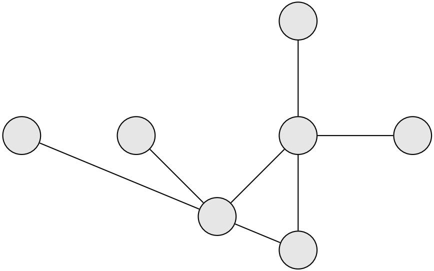

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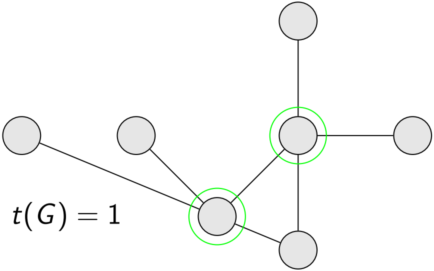

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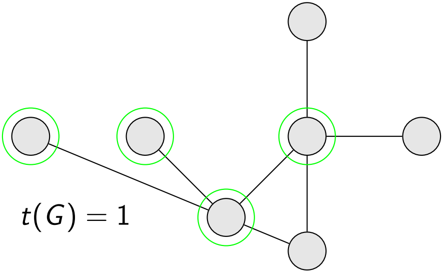

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- If W_i is not an independent set V_{i+1} as the union of the largest set of vertices that induces a subgraph with all degrees odd, of each W_i component
- Use lower bound of $f(n)$ to estimate number of steps in this procedure

Remark: The original proof was made when $f(n) \geq cn$ was not proven yet, so originally the bound was $\leq blog^2(n)$ with the remark, that if the conjecture is true, then the upper bound is tighter.

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Calculate $E(X)$ and $Var(X)$.
Using Chybyshev inequality bound $n \longrightarrow \inf P(X = 0) \longrightarrow 0$

Theorem

For odd n, almost every $G \in \mathcal{G}(n, \frac{1}{2})$ has a vertex partition $V(G) = V_1 V_2$ such that $G[V_1]$ has all degrees odd and $G[V_2]$ has exactly one vertex with even degree.

Proof is simmilar as for the previous theorem

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Question

For a random graph $G \in \mathcal{G}(n, \frac{1}{2})$, where n is even, what is $P(p(G) = 2)$

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Definition

$f_k(n) = \min\{f_k(G) : \delta(G) \geq 1 \text{ and } |G| = n\}$.

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Lemma

Let $k \geq 2$ be an integer. There exists $c(k) > 0$ such that for every bipartite graph G with $\delta(G) \geq 1$ there is a set $W \subset V(G)$ such that $|W| \geq c(k)|G|$ and $G[W]$ has all degrees congruent to 1 modulo k.

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Theorem

Let $k \geq 2$ be a fixed integer. Let G be a claw-free graph without isolated vertices with order n. Then $f_k(G) \geq (1 + o(1))\sqrt{\frac{n}{12}}$

Previous best result was proven by Y.Caro: $f_k(G) \geq ck(nlogn)^{\frac{1}{3}}$

Claw-graph