A note on polynomials and f-factors of graphs

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Definition

f-factor is a spanning subgraph H of G in which $d_H(v) \in f(v)$ for all $v \in V$.

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- If $|f(v)| = 1$ for every $v \in V(G)$, then there exists a necessary and sufficient condition for the existence of an f-factor of G (Tutte's f-factor theorem).
- No necessary and sufficient condition for an f-factor exists when we allow $|f(v)| \geq 2$, even when $|f(v)| = 2$ for all $v \in V$.
- On the other hand, if no two consecutive integers in $f(v)$ differ by more than two, then there is a necessary and sufficient condition for an f-factor (Lovász).

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$$|f(v)| > \lceil d(v)/2 \rceil$$

for every $v \in V$.
Then G has an f-factor.

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Let $g \in \mathbb{F}[X_1, X_2, ..., X_n]$ be a polynomial, and suppose the coefficient of the monomial $\prod_{i=1}^{n} X_i^{t_i}$ in g is non-zero, where $t_1 + t_2 + ... + t_n$ is the total degree of g.
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Therefore it is possible to assign to each $v \in V$ a set $E(v)$ of edges containing v such that $|E(v)| = |f(v)^c|$ and, for all distinct $u, v \in V$,

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Then

$$\prod_{v \in V} \prod_{e \in E(v)} X_e.$$

is a monomial of the required degree in g.

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Combinatorial Nullstellensatz

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Let $G = (V, E)$ be a graph, and let f satisfy

$$|E| > \sum_{v \in V} |f(v)^c \setminus \{0\}|$$

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Partial f-factor of a graph $G = (V, E)$ is an $\tilde{f}$-factor of G where $\tilde{f}(v) = f(v) \cup \{0\}$ for all $v \in V$.

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Partial f-factor is **non-trivial** if it is non-empty.

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By the inequality of the theorem, the total degree of the first term in g is

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so the largest degree monomial in g is precisely $(-1)^{|E|+1} \prod_{e \in E} X_e$.

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Bibliography

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