Dimension and cut vertices: an application of Ramsey theory

William T. Trotter, Bartosz Walczak and Ruidong Wang

2017

Definition. *Partially ordered set* (*poset*)

Definition. *Partially ordered set (poset)*

$$P = (X, \leq_P)$$

Definition. *Partially ordered set* (*poset*)

$P = (X, \leq_P)$

X - the *ground set*, or *set of elements*

Definition. *Partially ordered set* (*poset*)

$P = (X, \leq_P)$

X - the *ground set*, or *set of elements*

$\leq_P$ - reflexive, antisymmetric and transitive relation

Definition. *Partially ordered set* (*poset*)

$P = (X, \leq_P)$

X - the *ground set*, or *set of elements*

$\leq_P$ - reflexive, antisymmetric and transitive relation

$\leq_P \subset X \times X$

Definition. *Partially ordered set* (*poset*)

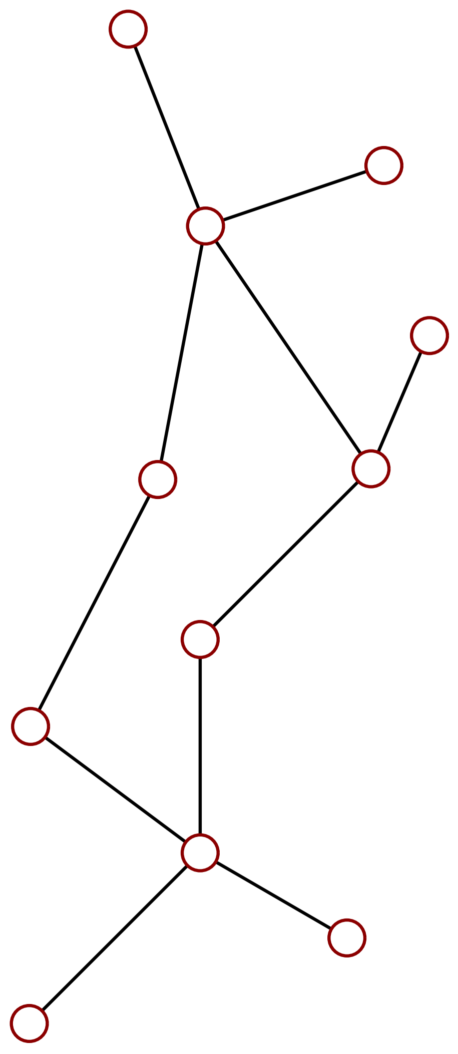
$P = (X, \leq_P)$

X - the *ground set*, or *set of elements*

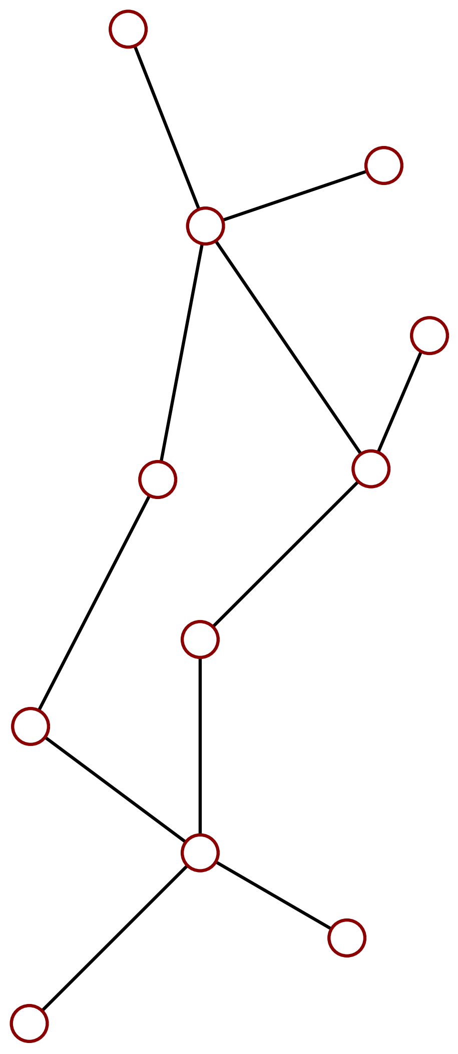
$\leq_P$ - reflexive, antisymmetric and transitive relation

$\leq_P \subset X \times X$

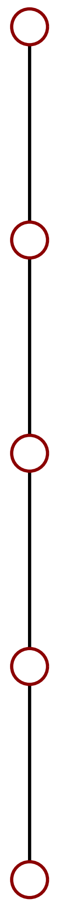
Intuition: posets are sets with some inequalities between elements



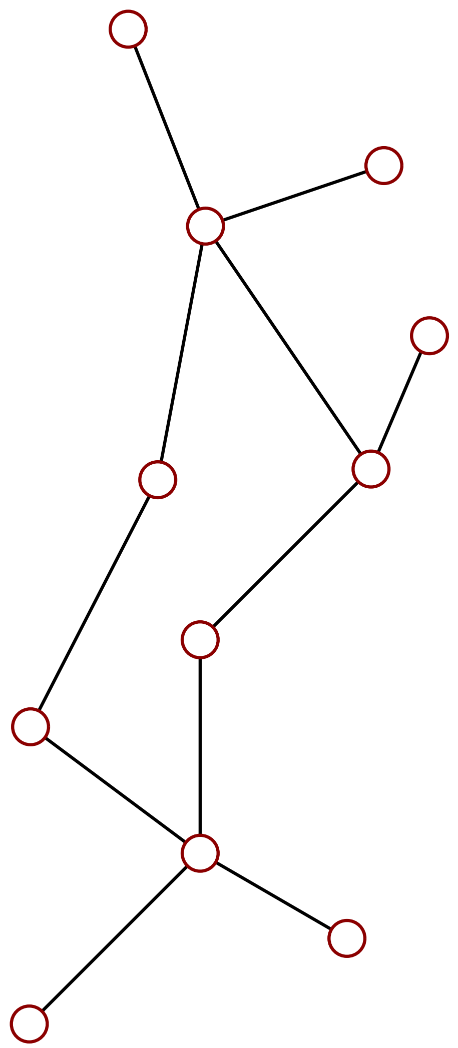
poset



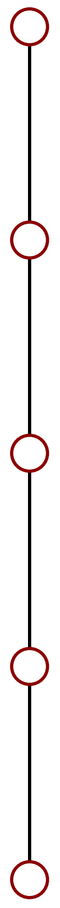
poset



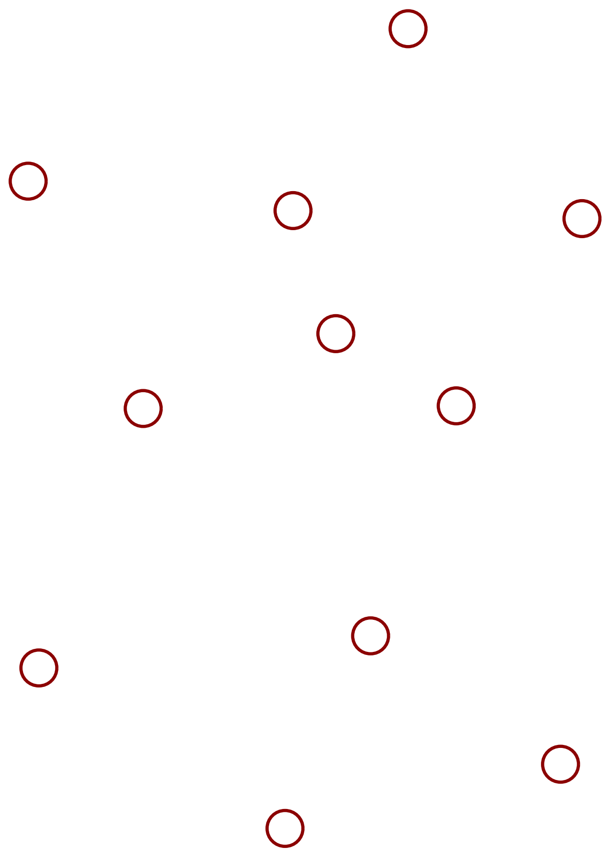
chain



poset



chain



antichain

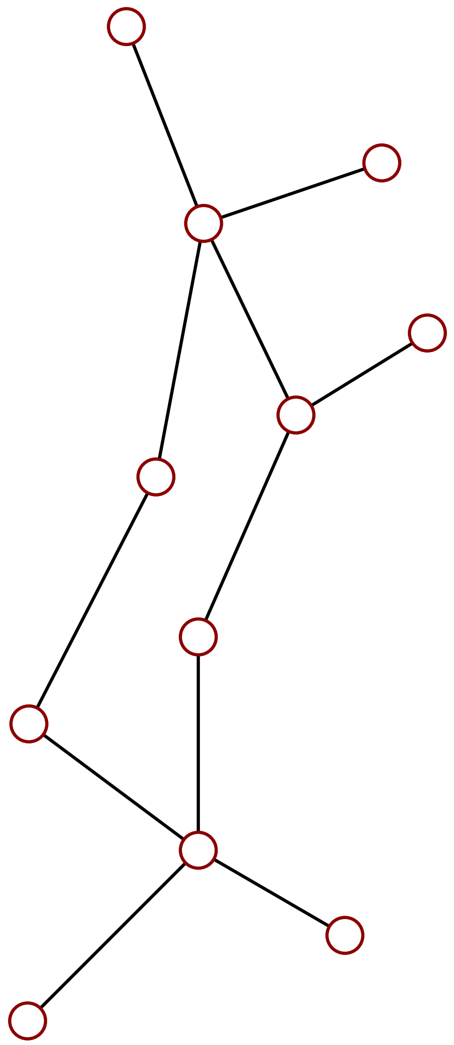
Definition. *Linear extension* of poset P

Definition. *Linear extension* of poset P

Chain that extends P

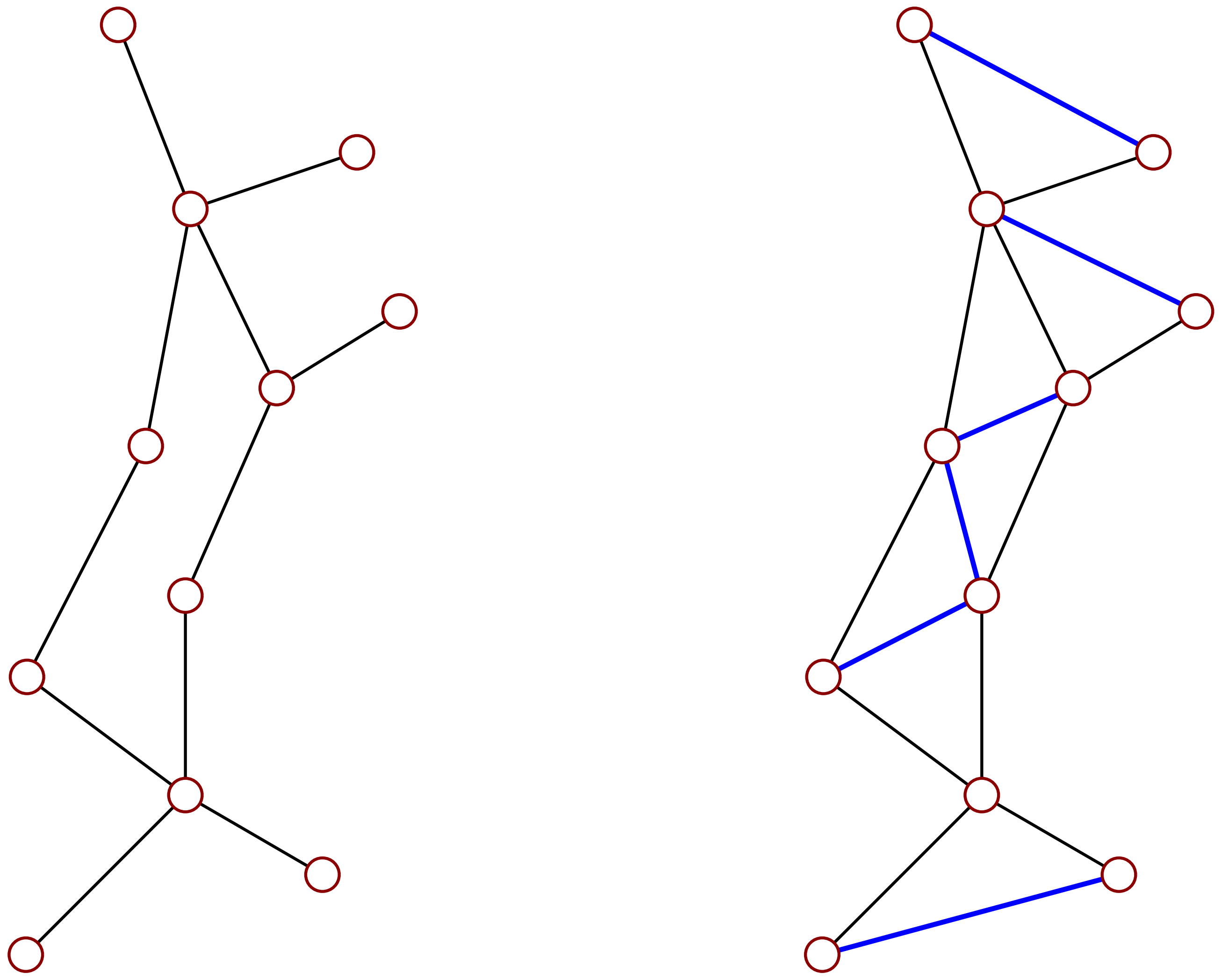
Definition. *Linear extension* of poset P

Chain that extends P



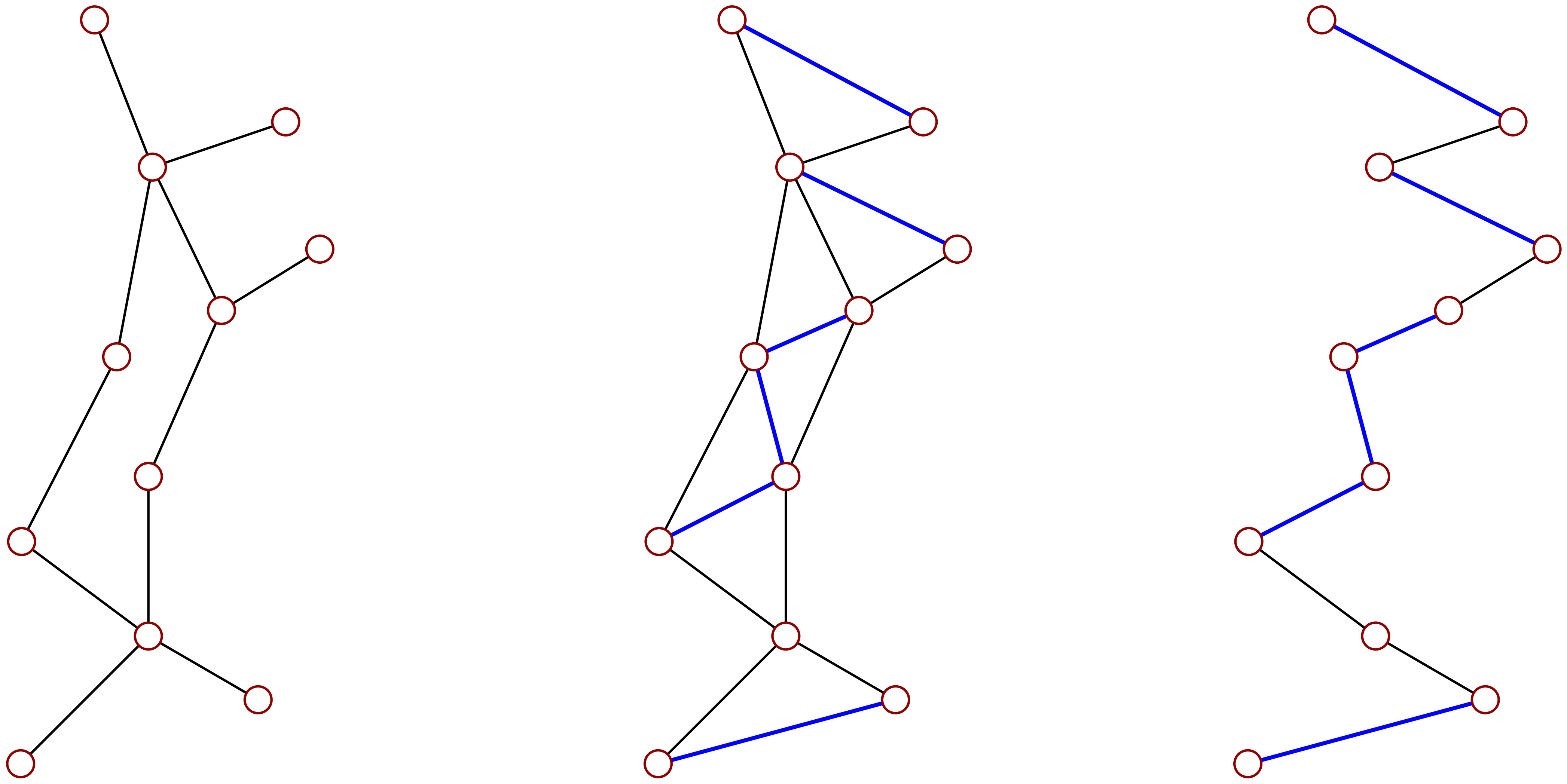
Definition. *Linear extension* of poset P

Chain that extends P



Definition. *Linear extension* of poset P

Chain that extends P



Definition. *Realizer* of poset P

Definition. *Realizer* of poset P

Set $\{L_1, L_2, ..., L_d\}$ of linear extensions of P such that

$$L_1 \cap L_2 \cap ... \cap L_d = P$$

Definition. *Realizer* of poset P

Set $\{L_1, L_2, ..., L_d\}$ of linear extensions of P such that

$$L_1 \cap L_2 \cap ... \cap L_d = P$$

Definition. *Dushnik-Miller dimension* (or simply *dimension*) of poset P

Definition. *Realizer* of poset P

Set $\{L_1, L_2, ..., L_d\}$ of linear extensions of P such that

$$L_1 \cap L_2 \cap ... \cap L_d = P$$

Definition. *Dushnik-Miller dimension* (or simply *dimension*) of poset P

The minimum possible size of realizer of P

Definition. *Realizer* of poset P

Set $\{L_1, L_2, ..., L_d\}$ of linear extensions of P such that

$$L_1 \cap L_2 \cap ... \cap L_d = P$$

Definition. *Dushnik-Miller dimension* (or simply *dimension*) of poset P

The minimum possible size of realizer of P

We denote it by $dim(P)$

$dim(P)$ is equal to minimum d such that there is an embedding $P \longrightarrow \mathbb{R}^d$

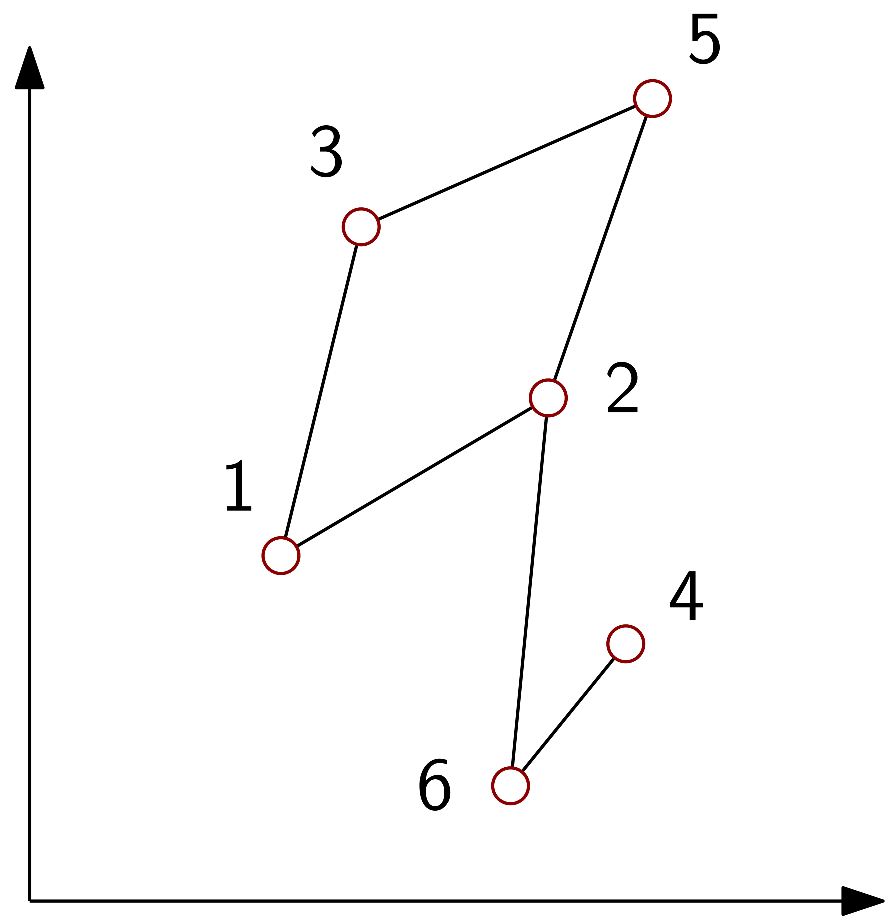
$dim(P)$ is equal to minimum d such that there is an $\boxed{\text{embedding}}$ $P \longrightarrow \mathbb{R}^d$

$f : x \longrightarrow (x_1, x_2, ..., x_d)$ s.t. $x \leq_P y \iff (x_1 \leq y_1) \wedge (x_2 \leq y_2) \wedge ... \wedge (x_d \leq y_d)$

$dim(P)$ is equal to minimum d such that there is an $\boxed{\text{embedding}}$ $P \longrightarrow \mathbb{R}^d$

$$f : x \longrightarrow (x_1, x_2, ..., x_d) \text{ s.t. } x \leq_P y \iff (x_1 \leq y_1) \wedge (x_2 \leq y_2) \wedge ... \wedge (x_d \leq y_d)$$

$$(1,3), (1,5), (1,2), (3,5), (2,5), (6,2), (6,4) \xrightarrow{f}$$



Computer science corner. What is the complexity of determining whether dimension is at most k?

Computer science corner. What is the complexity of determining whether dimension is at most k?

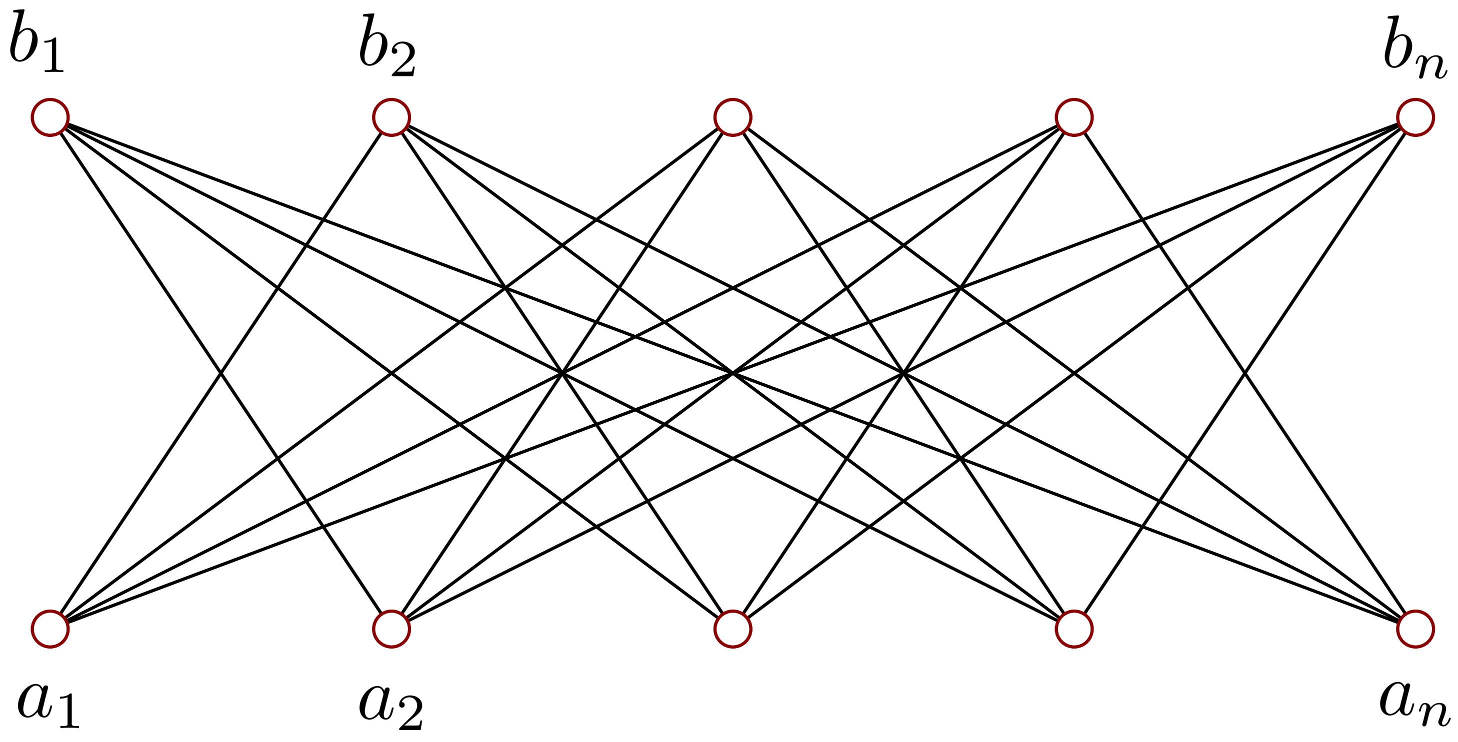
- P for $k \leq 2$ - reduction to *recognition of transitively orientable graphs*

Computer science corner. What is the complexity of determining whether dimension is at most k?

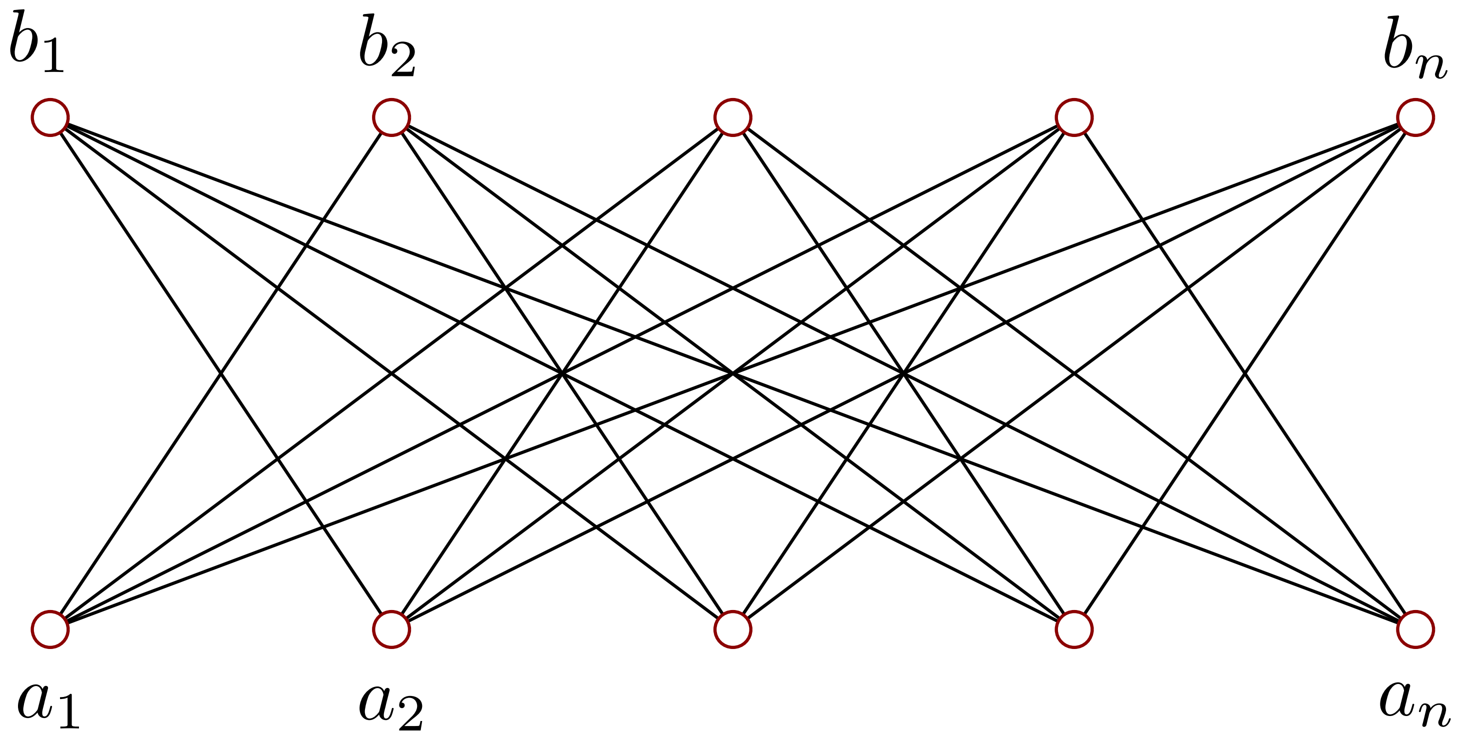
- P for $k \leq 2$ - reduction to *recognition of transitively orientable graphs*
- NP-complete for $k \geq 3$ - reduction from *chromatic number 3* [M. Yannakakis, 1982]

Problem. How big can dimension be?

Problem. How big can dimension be?



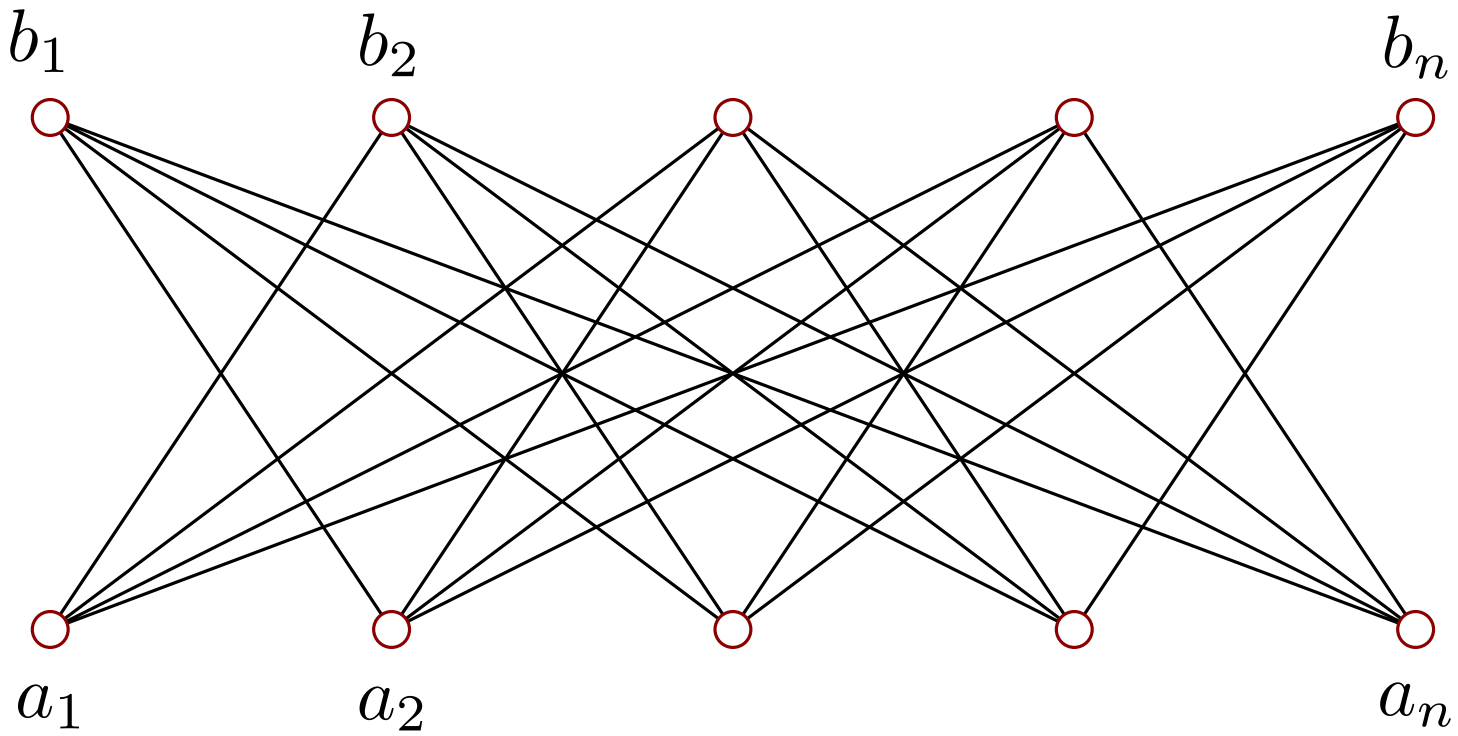
Problem. How big can dimension be?



Standard example S_n

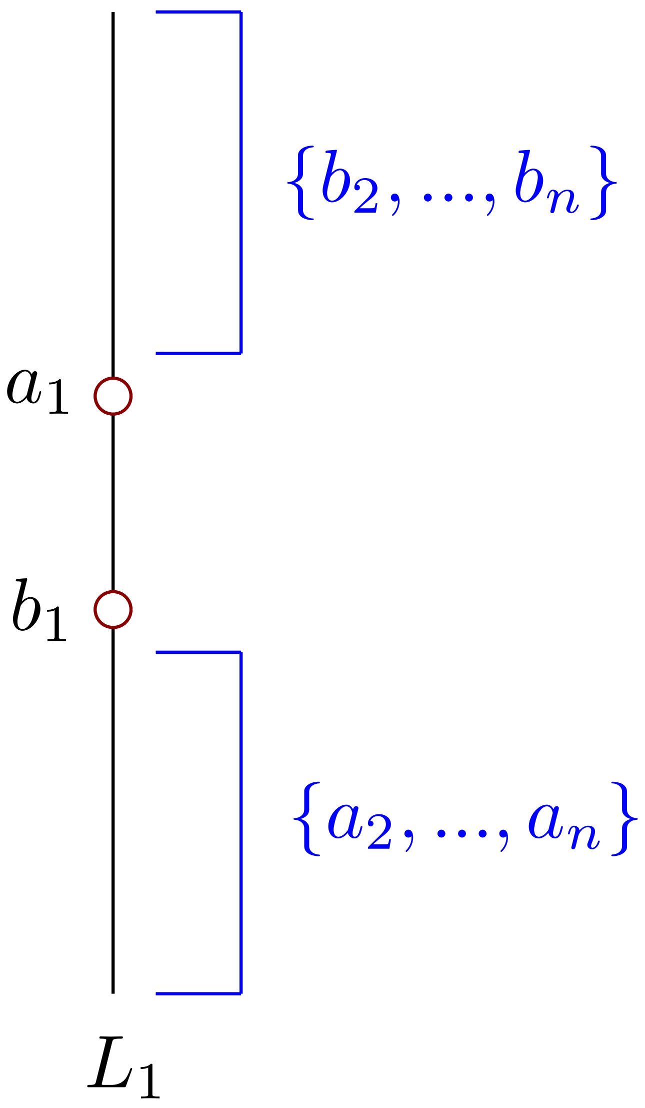
$$dim(S_n) = n$$

Problem. How big can dimension be?

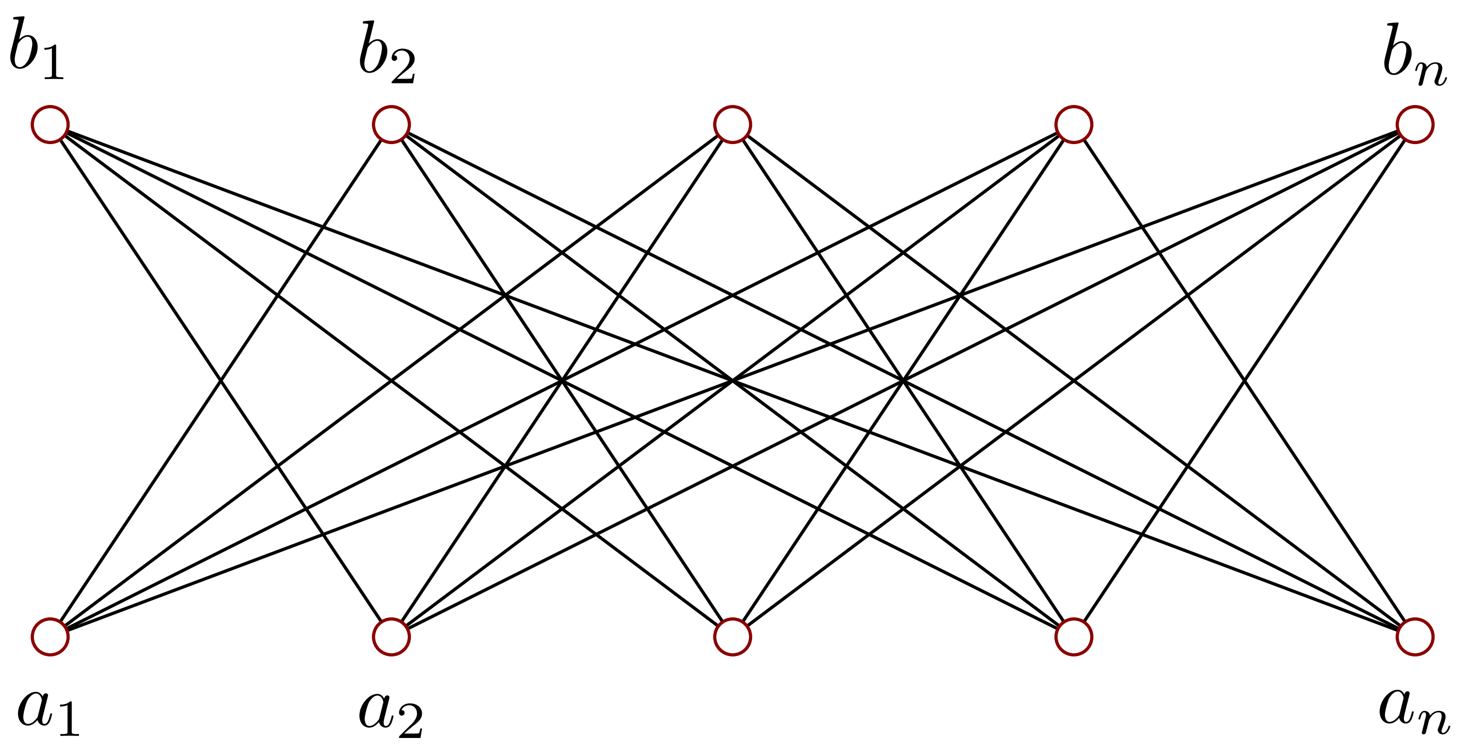


Standard example S_n

$$dim(S_n) = n$$

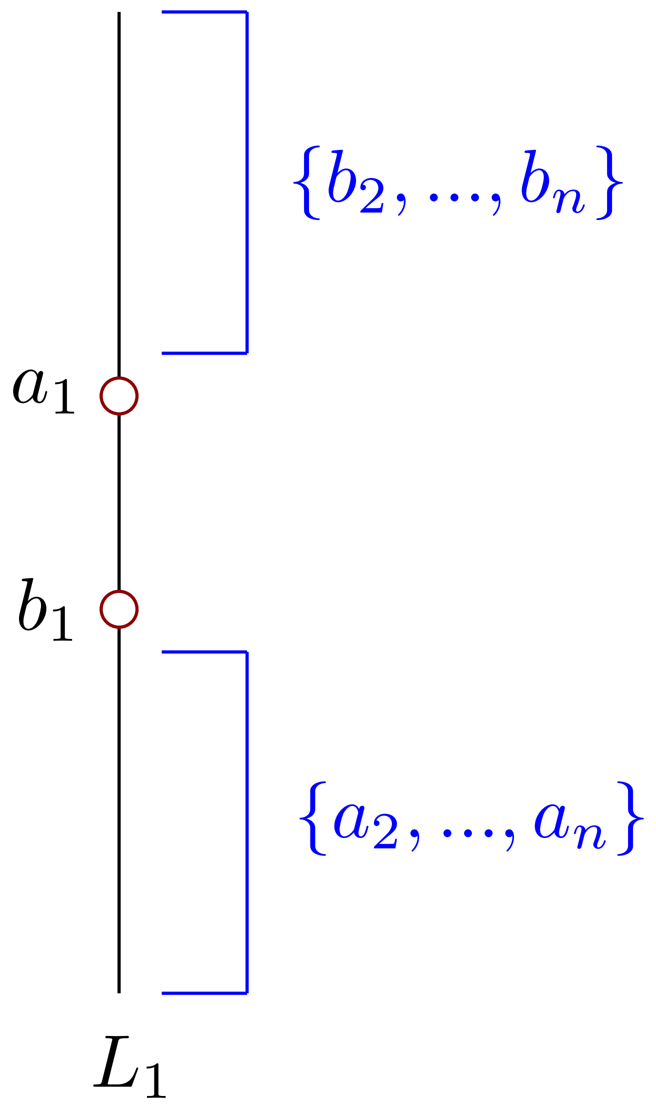


Problem. How big can dimension be?



Standard example S_n

$$dim(S_n) = n$$



This is the worst case. Generally for $|P| \geq 4$, $dim(P) \leq |P|/2$

Problem. What makes the dimension large?

Problem. What makes the dimension large?

- cover graph is a forest $\implies dim(P) \leq 3$ (and this bound is best possible) [Moore, Trotter, 1977]

Problem. What makes the dimension large?

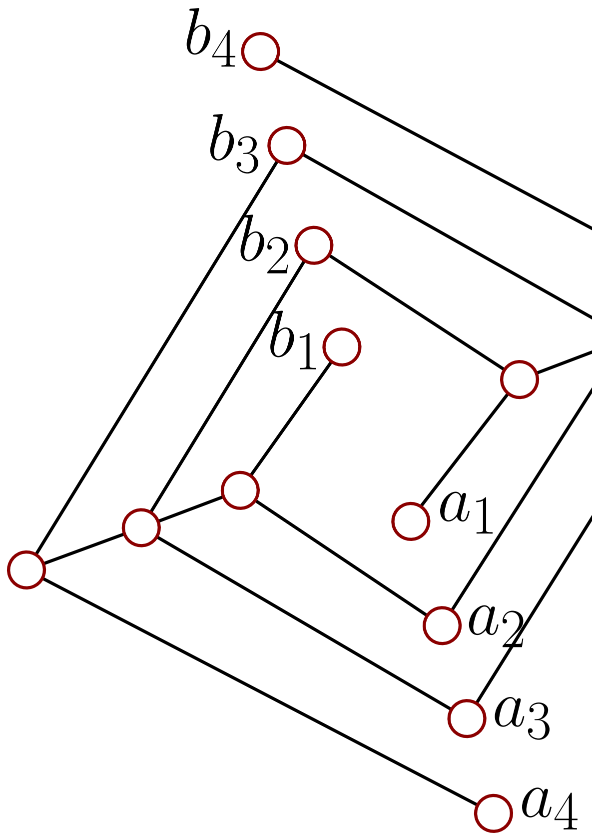
- cover graph is a forest $\implies dim(P) \leq 3$ (and this bound is best possible) [Moore, Trotter, 1977]

  Idea: Maybe if cover graph is "sparse" in some measure, the dimension is always small?

Problem. What makes the dimension large?

- cover graph is a forest $\implies$ $dim(P) \leq 3$ (and this bound is best possible) [Moore, Trotter, 1977]

  Idea: Maybe if cover graph is "sparse" in some measure, the dimension is always small?

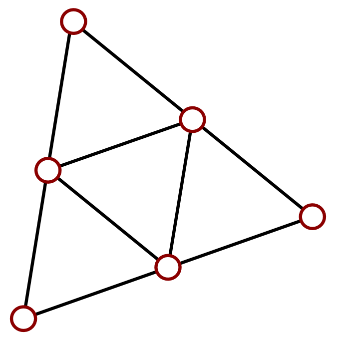


Kelly's example. Posets with planar diagrams and arbitrarily large dimension.

Then maybe something stronger?

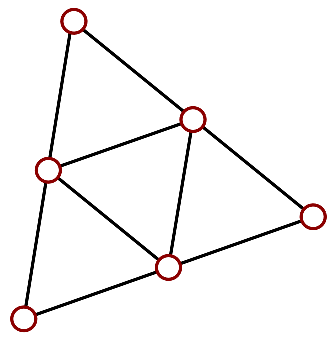
Then maybe something stronger?

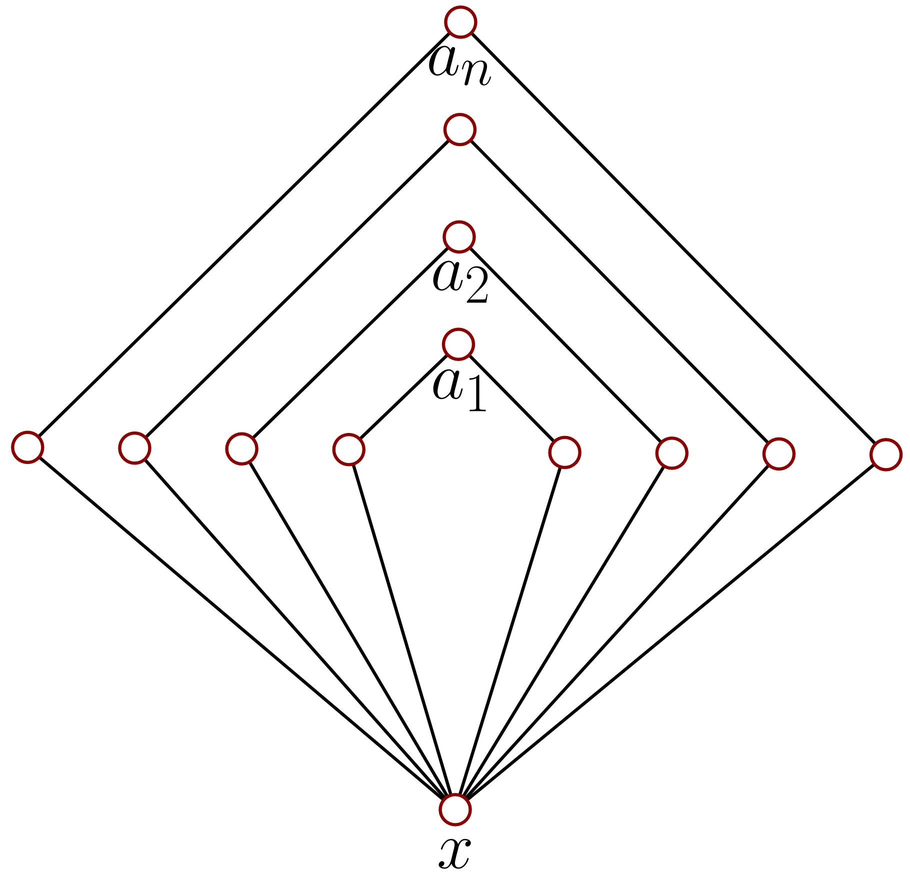
- cover graph is outerplanar $\implies dim(P) \leq 4$ (and the bound is best possible) [Felsner, Trotter, Wiechert, 2015]



Then maybe something stronger?

- cover graph is outerplanar $\implies dim(P) \leq 4$ (and the bound is best possible) [Felsner, Trotter, Wiechert, 2015]

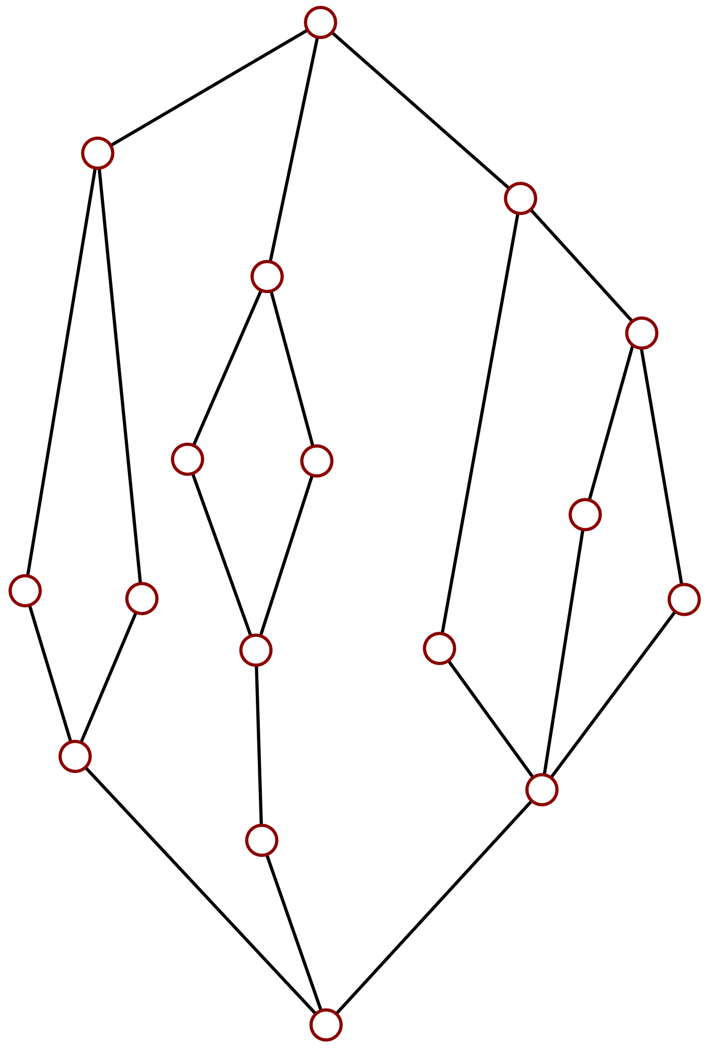




$dim(P) = 4$ for $n \geq 17$

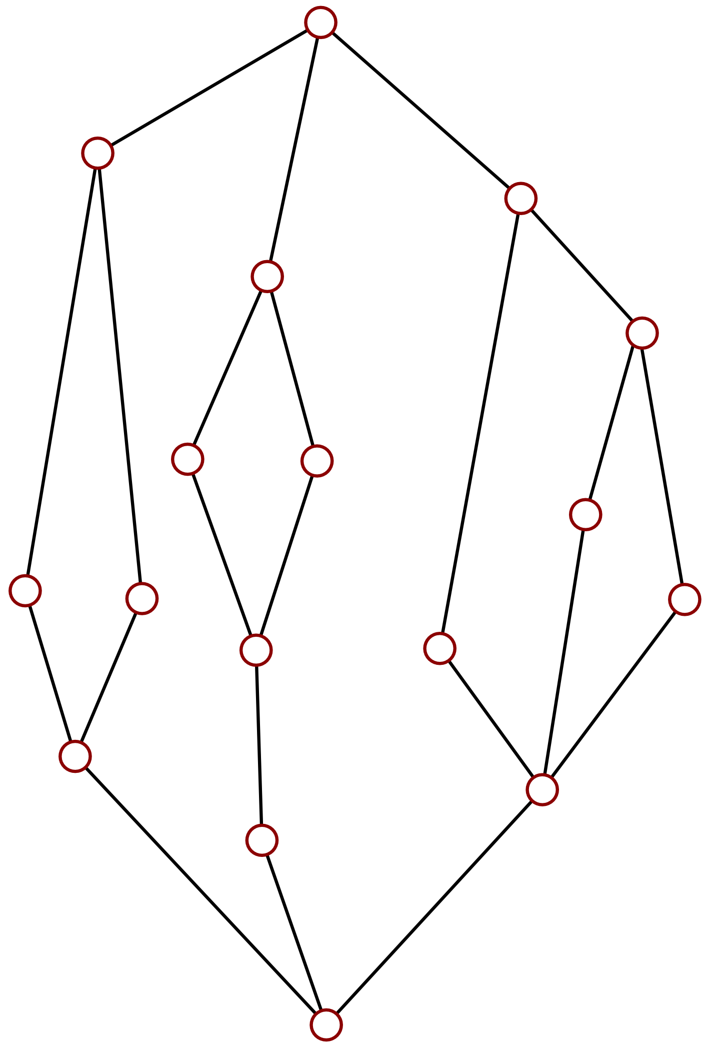
What about other sparsity measures? Maybe tree-width?

What about other sparsity measures? Maybe tree-width?



- cover graph has tree-width at most 2 $\implies dim(P) \leq 1276$ [Joret, Micek, Trotter, Wang, Wiechert, 2014]

What about other sparsity measures? Maybe tree-width?



- cover graph has tree-width at most 2 $\implies$ $dim(P) \leq \cancel{1276}$
  [Joret, Micek, Trotter, Wang, Wiechert, 2014]

- cover graph has tree-width at most 2 $\implies$ $dim(P) \leq 12$
  [Seweryn, 2020]

Alright. What about... height?

Alright. What about... height?

- There is a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that if height$(P) \leq h$ and P has planar cover graph, then

$$dim(P) \leq f(h)$$

[Streib, Trotter, 2014]

Alright. What about... height?

- There is a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that if height$(P) \leq h$ and P has planar cover graph, then

$$dim(P) \leq f(h) \qquad f = \mathcal{O}(4^{h^3})$$

[Streib, Trotter, 2014]

Alright. What about... height?

- There is a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that if height$(P) \leq h$ and P has planar cover graph, then

$$dim(P) \leq f(h) \qquad f = \mathcal{O}(4^{h^3})$$

[Streib, Trotter, 2014]

- There is a function $f : \mathbb{N}^2 \longrightarrow \mathbb{N}$ such that if $height(P) \leq h$ and tree-width $\leq t$, then

$$dim(P) \leq f(h, t)$$

[Joret, Micek, Milans, Trotter, Walczak, Wang, 2016]

- There is a function $f : \mathbb{N}^2 \longrightarrow \mathbb{N}$ such that if $height(P) \leq h$ and cover graph does not contain K_t as a minor, then

$$dim(P) \leq f(h, t)$$

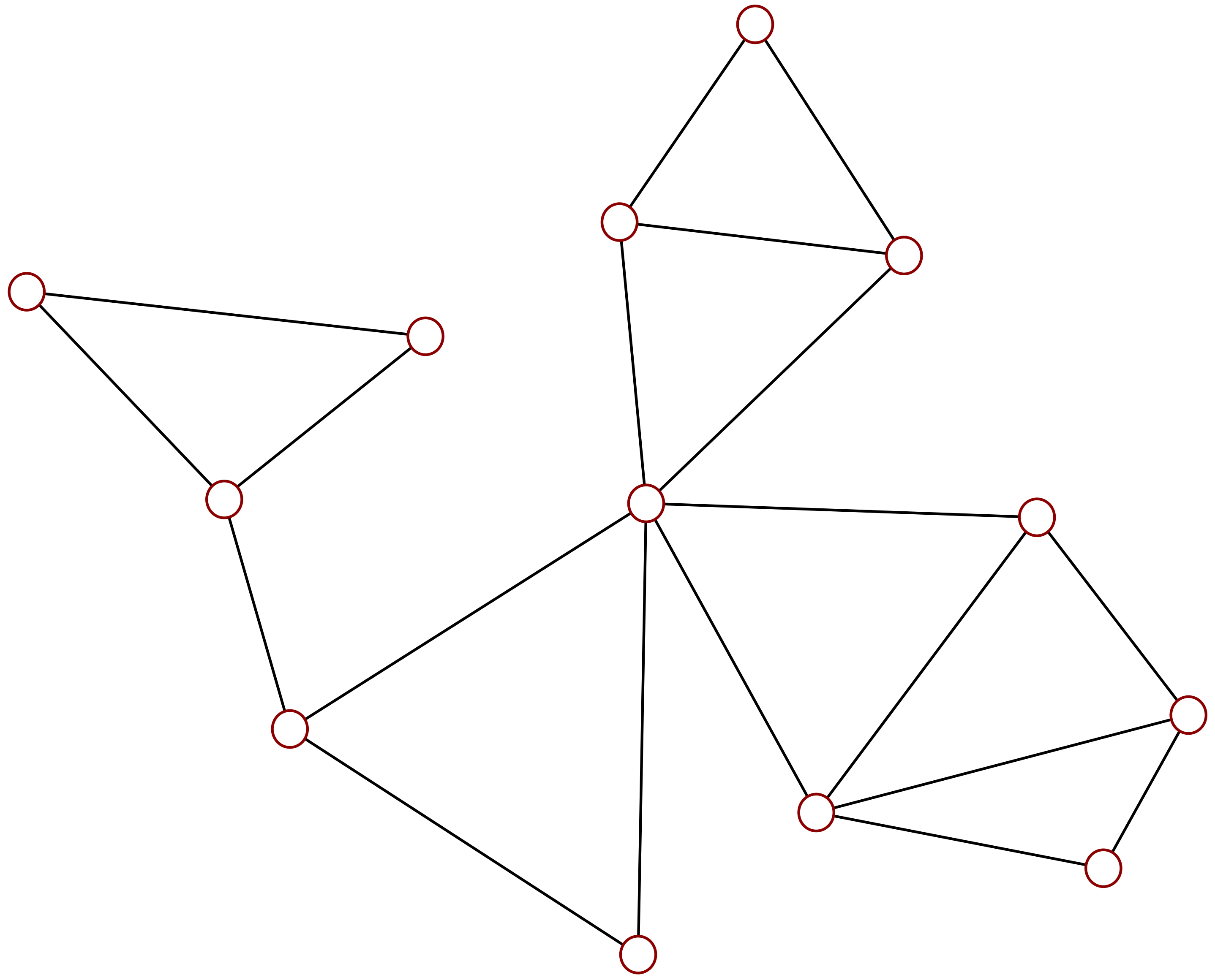
[Walczak, 2017]

- There is a function $f : \mathbb{N}^2 \longrightarrow \mathbb{N}$ such that if $height(P) \leq h$ and cover graph does not contain K_t as a minor, then

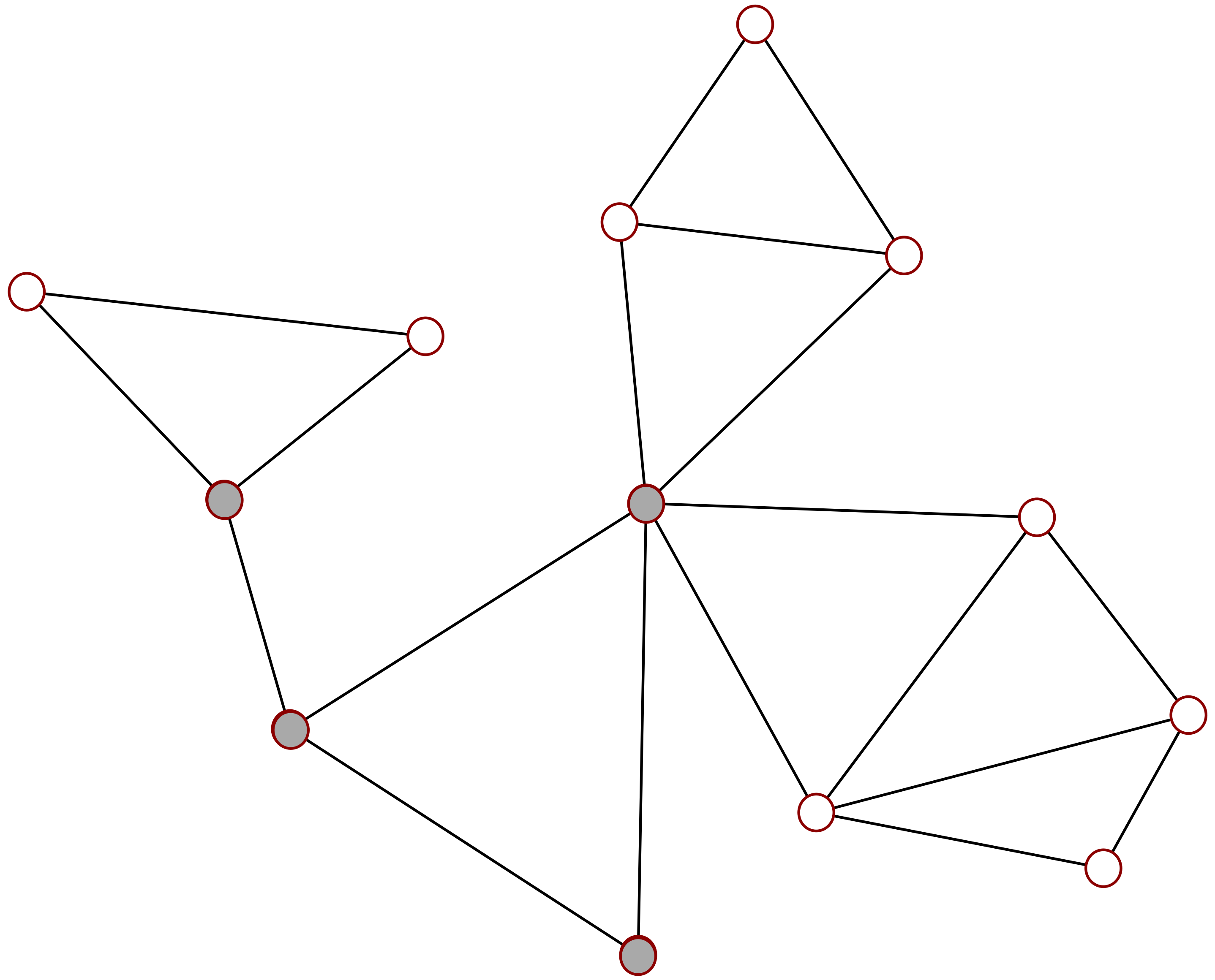
$$dim(P) \leq f(h, t)$$

[Walczak, 2017]

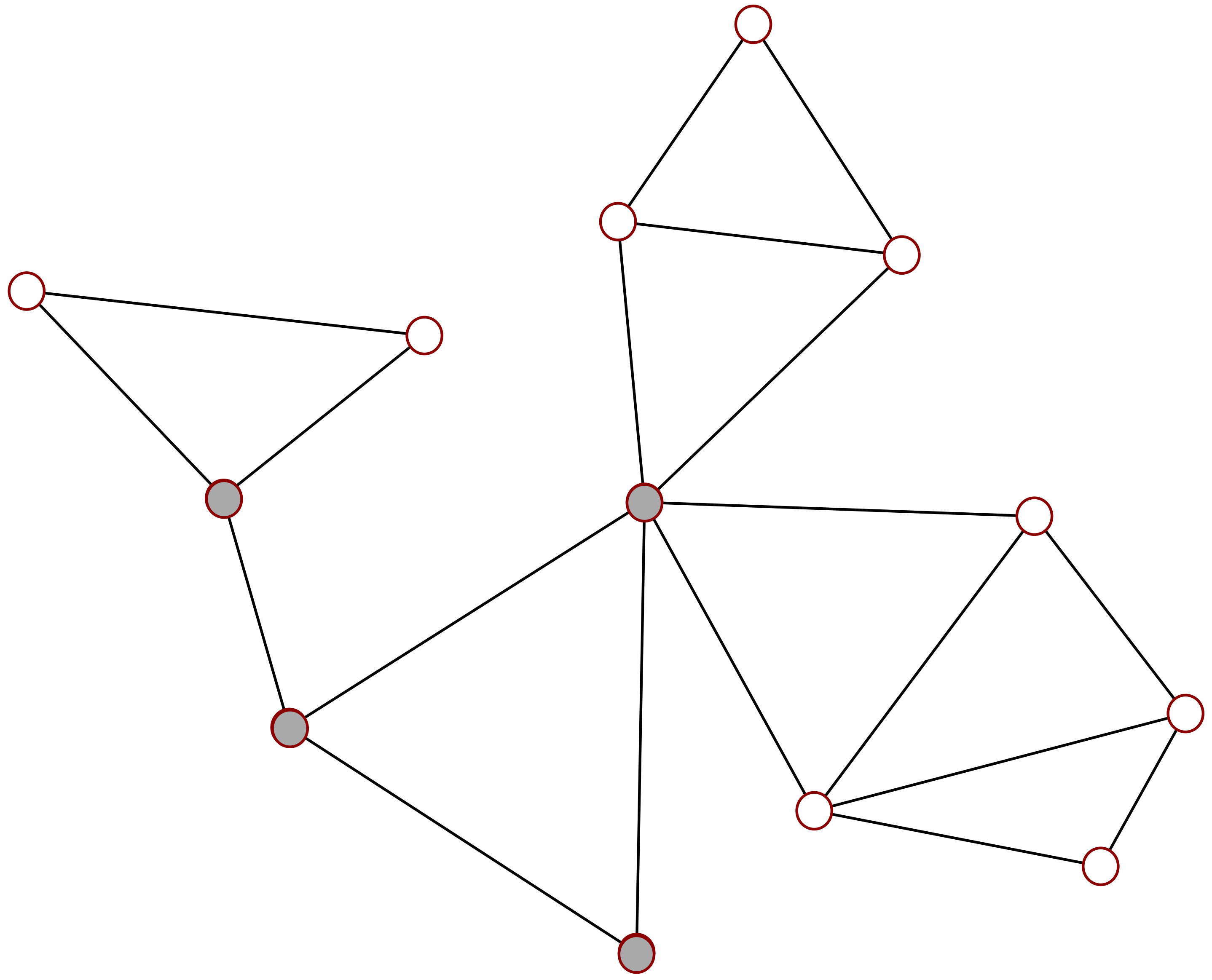
Now let's move to our today's topic...



Definition. An *articulation point* is a vertex that disconnects the graph when removed.

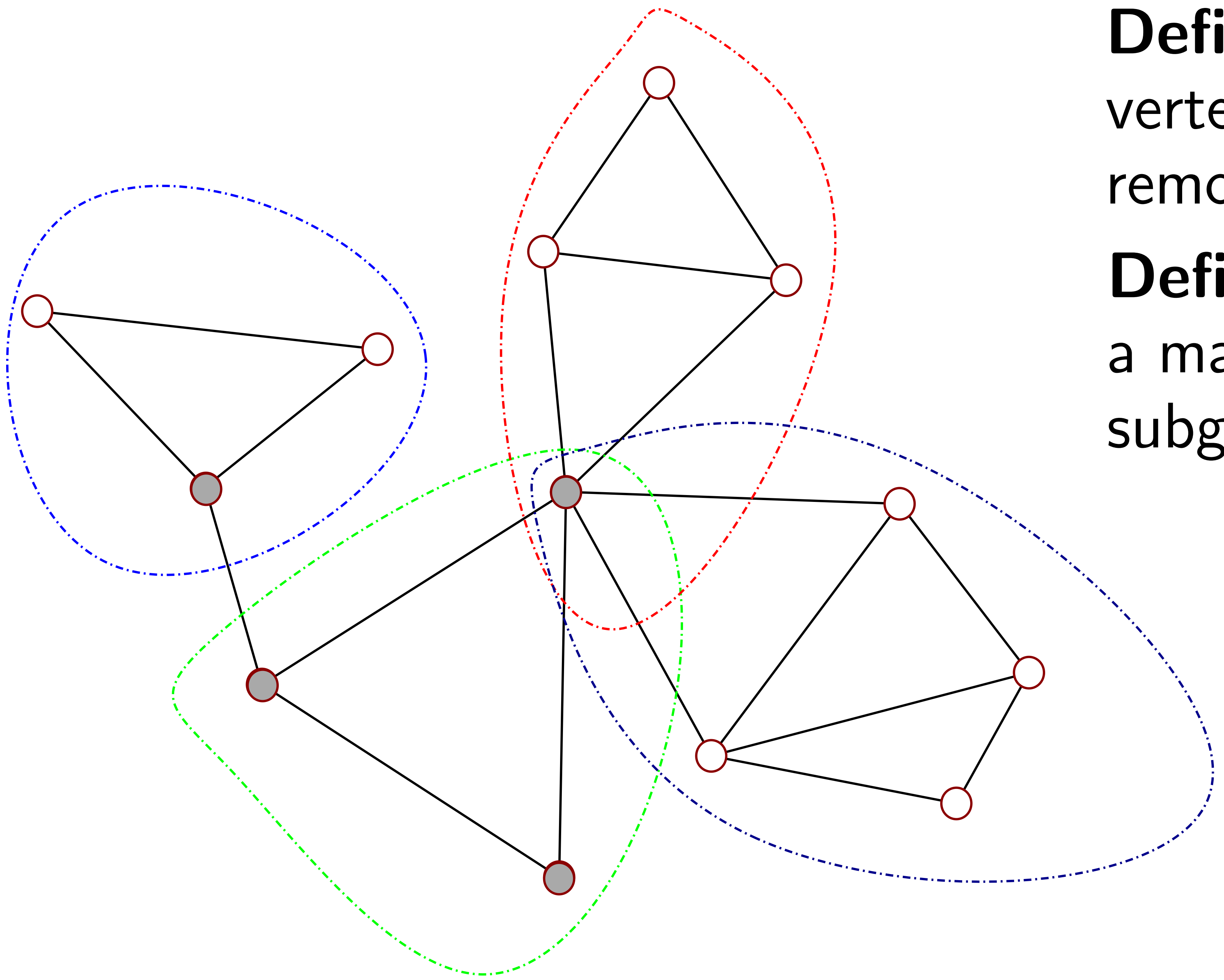


Definition. An *articulation point* is a vertex that disconnects the graph when removed.



Definition. An *articulation point* is a vertex that disconnects the graph when removed.

Definition. A *block* in a graph G, is a maximal induced 2-vertex-connected subgraph $H \subseteq G$.



Definition. An *articulation point* is a vertex that disconnects the graph when removed.

Definition. A *block* in a graph G, is a maximal induced 2-vertex-connected subgraph $H \subseteq G$.

Let $Inc(P)$ be the set of incomparable pairs (ordered!) of elements in P.

Let $Inc(P)$ be the set of incomparable pairs (ordered!) of elements in P.

We say that $R \subseteq Inc(P)$ is reversible, if there is a linear extension L of P s.t. for every $(x, y) \in R$ we have $x \leq y$ in L.

Let $Inc(P)$ be the set of incomparable pairs (ordered!) of elements in P.

We say that $R \subseteq Inc(P)$ is reversible, if there is a linear extension L of P s.t. for every $(x, y) \in R$ we have $x \leq y$ in L.

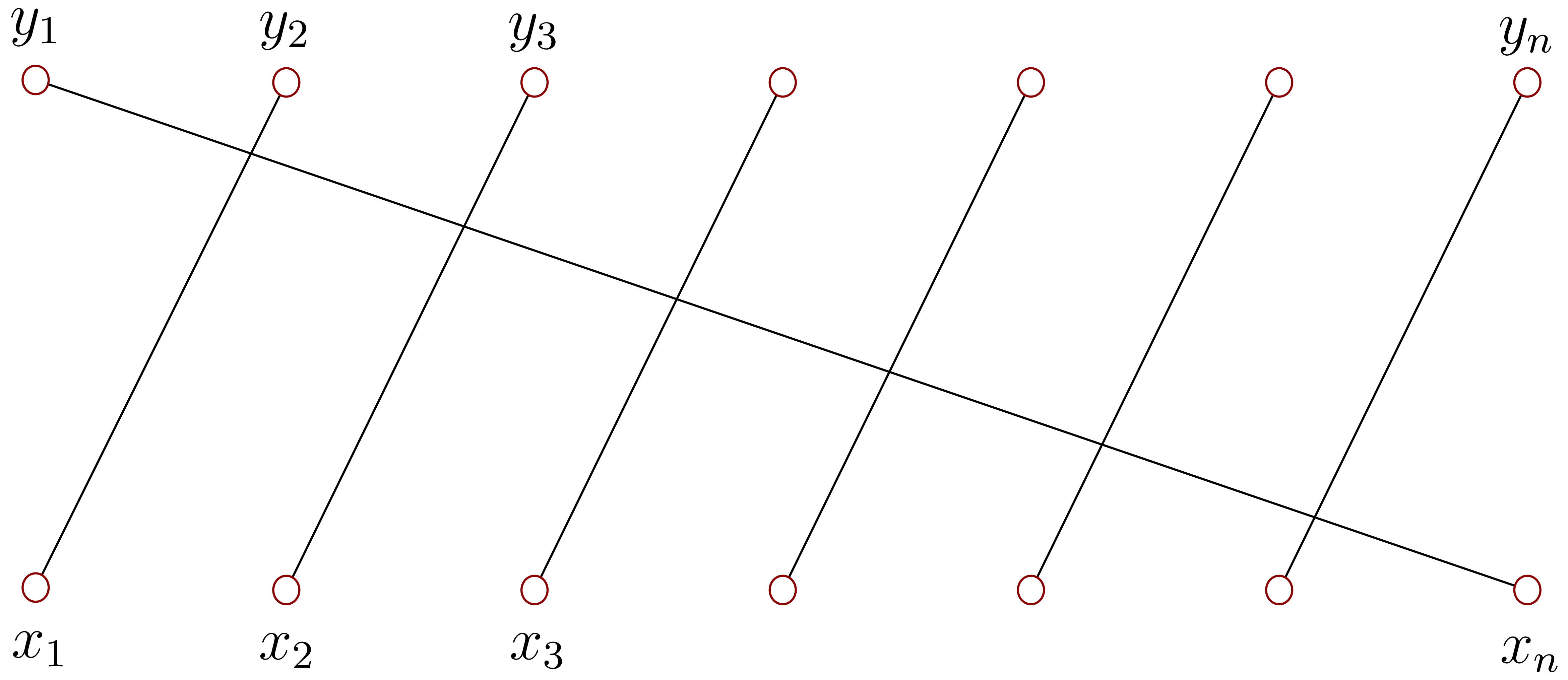
Observation. $dim(P)$ is the minimum number d s.t. there exist d reversible sets $R_1 \cup R_2 \cup ... \cup R_d = Inc(P)$.

Let $Inc(P)$ be the set of incomparable pairs (ordered!) of elements in P.

We say that $R \subseteq Inc(P)$ is reversible, if there is a linear extension L of P s.t. for every $(x, y) \in R$ we have $x \leq y$ in L.

Observation. $dim(P)$ is the minimum number d s.t. there exist d reversible sets $R_1 \cup R_2 \cup ... \cup R_d = Inc(P)$.

Useful fact. R is reversible $\iff$ R does not contain *alternating cycle*.



alternating cycle on incomparable pairs $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$

Main theorem of the article

Main theorem of the article

For every $d \geq 1$, if P is a poset and every block in P has dimension at most d, the the dimension of P is at most $d + 2$. Futhermore, this inequality is best possible.
[Trotter, Walczak, Wang, 2017]

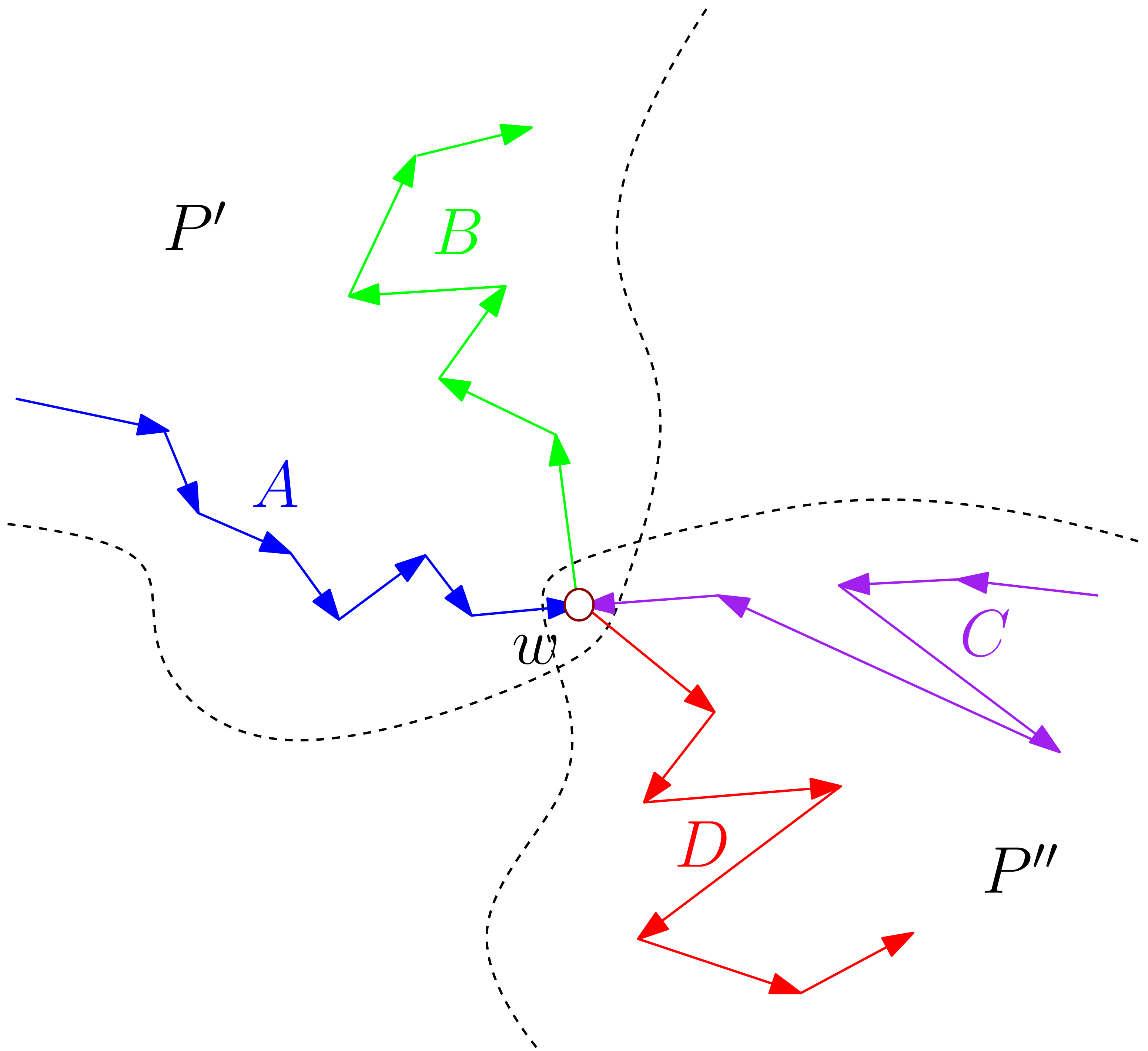
Main theorem of the article

For every $d \geq 1$, if P is a poset and every block in P has dimension at most d, the the dimension of P is at most $d+2$. Futhermore, this inequality is best possible.
[Trotter, Walczak, Wang, 2017]

Proof sketch:

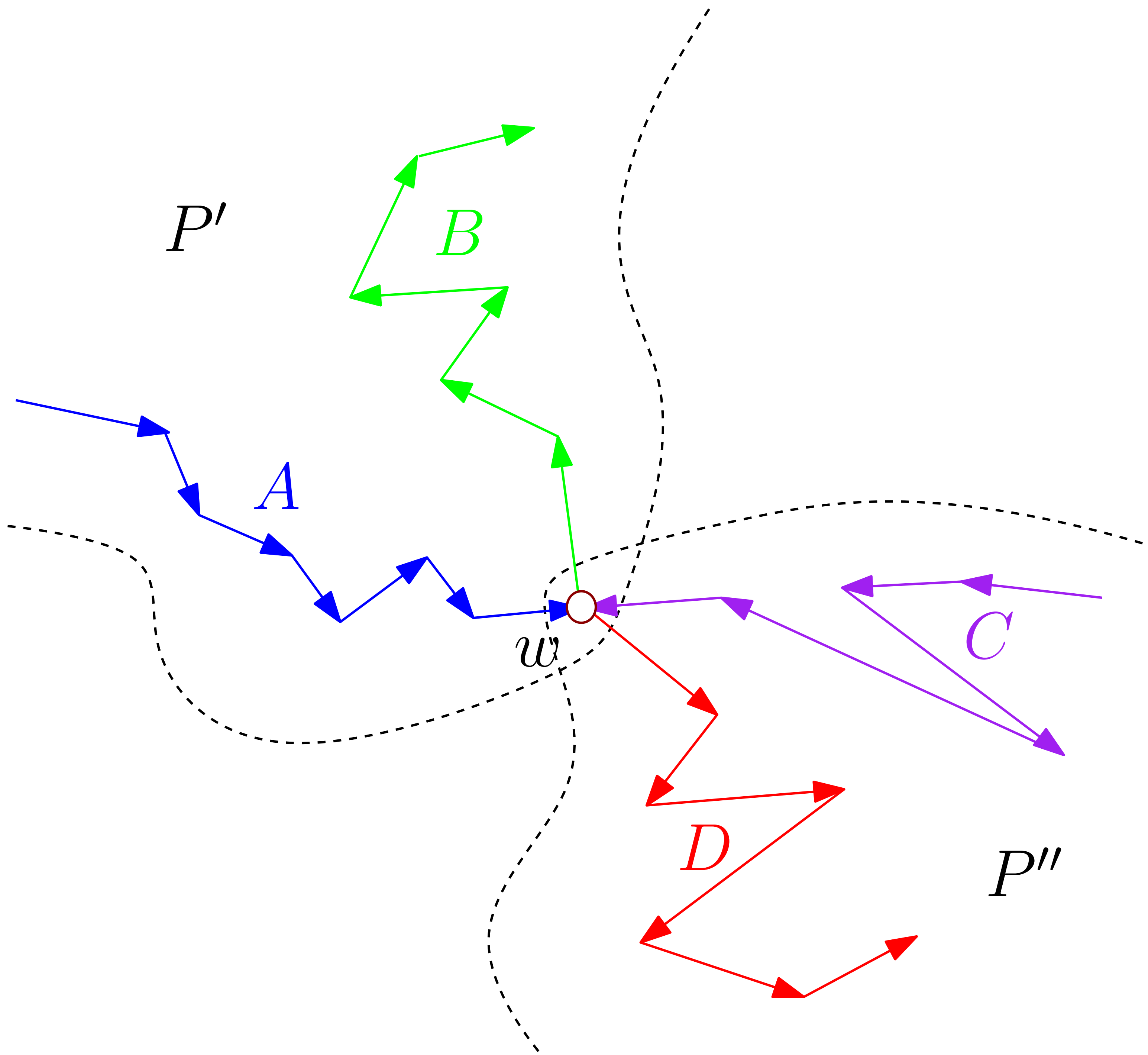
Let's define a tool we will be using: **the merge rule**

Let's define a tool we will be using: **the merge rule**

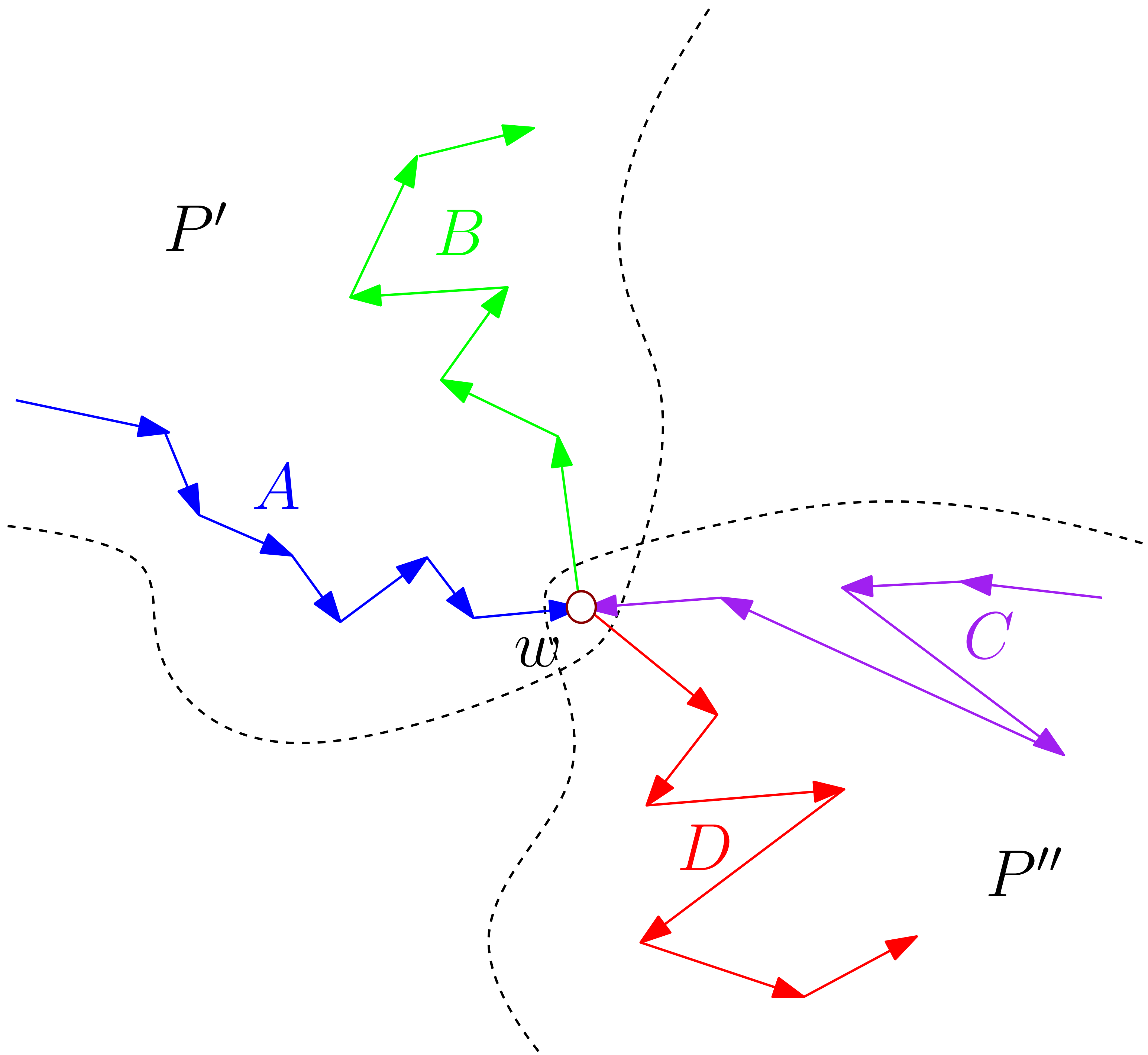


Let's define a tool we will be using: **the merge rule**

w separates subposets P' and P''



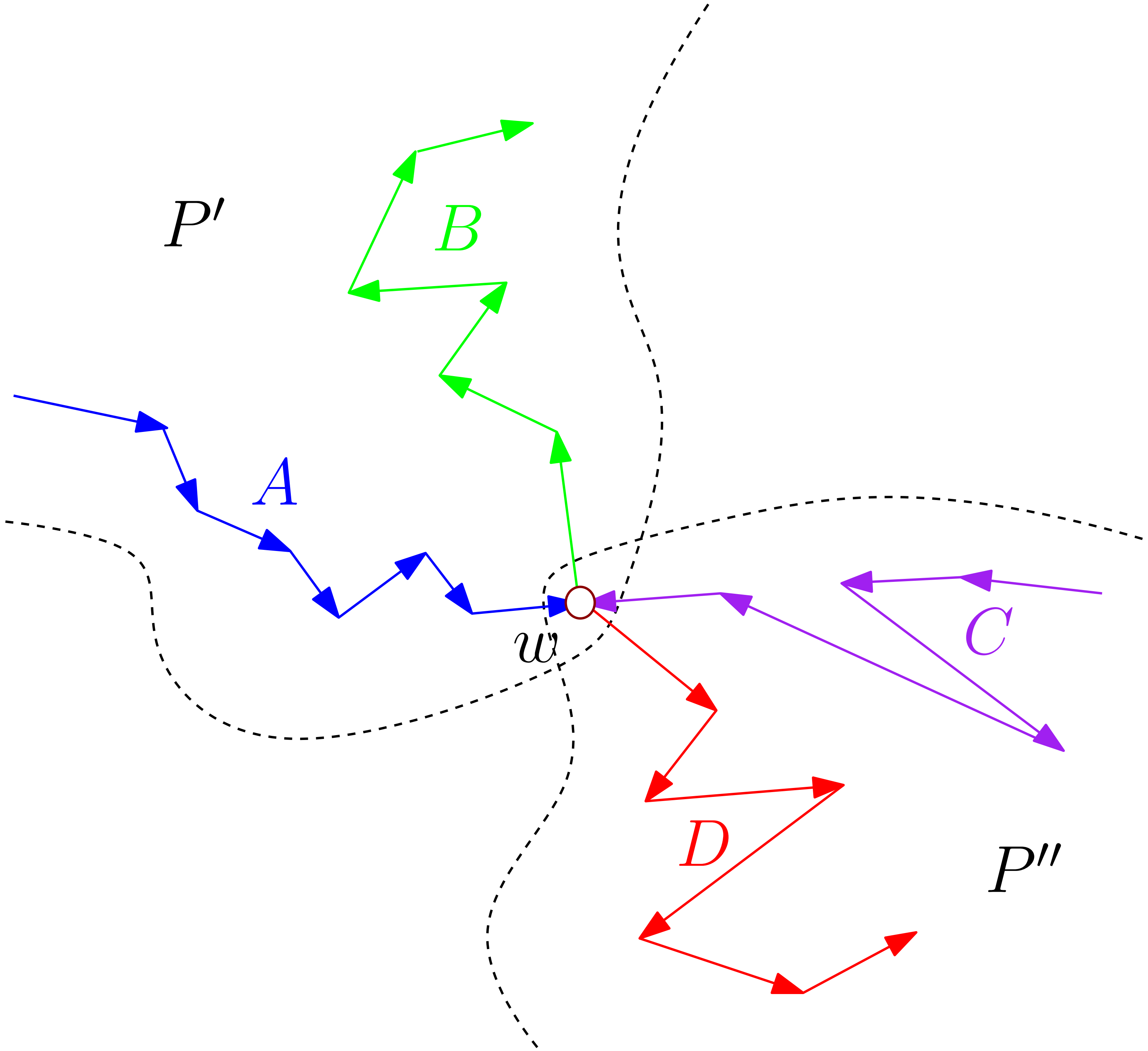
Let's define a tool we will be using: **the merge rule**



w separates subposets P' and P''

$[A < w < B]$ and $[C < w < D]$ are linear extensions of P' and P'' respectively

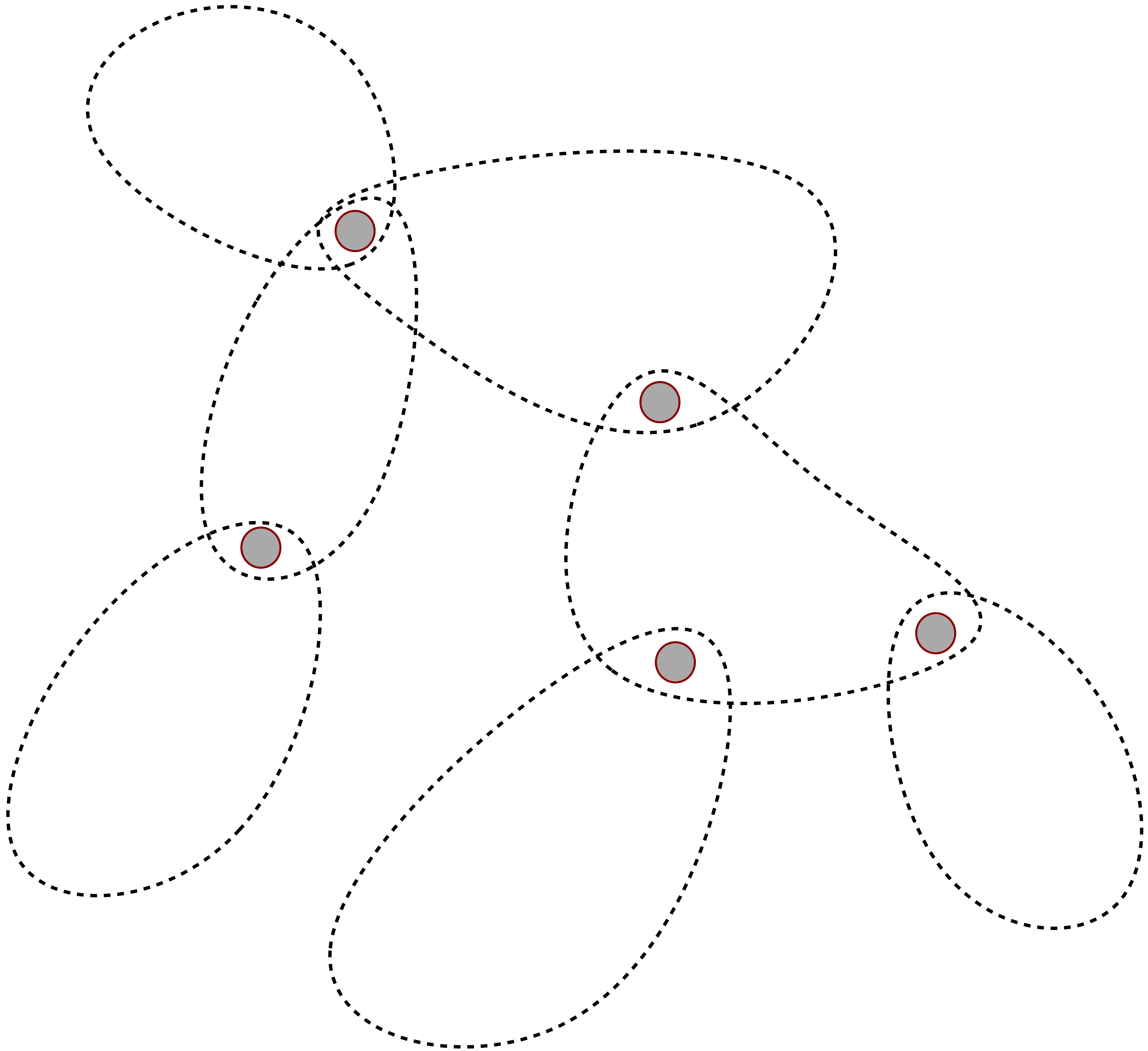
Let's define a tool we will be using: **the merge rule**



w separates subposets P' and P''

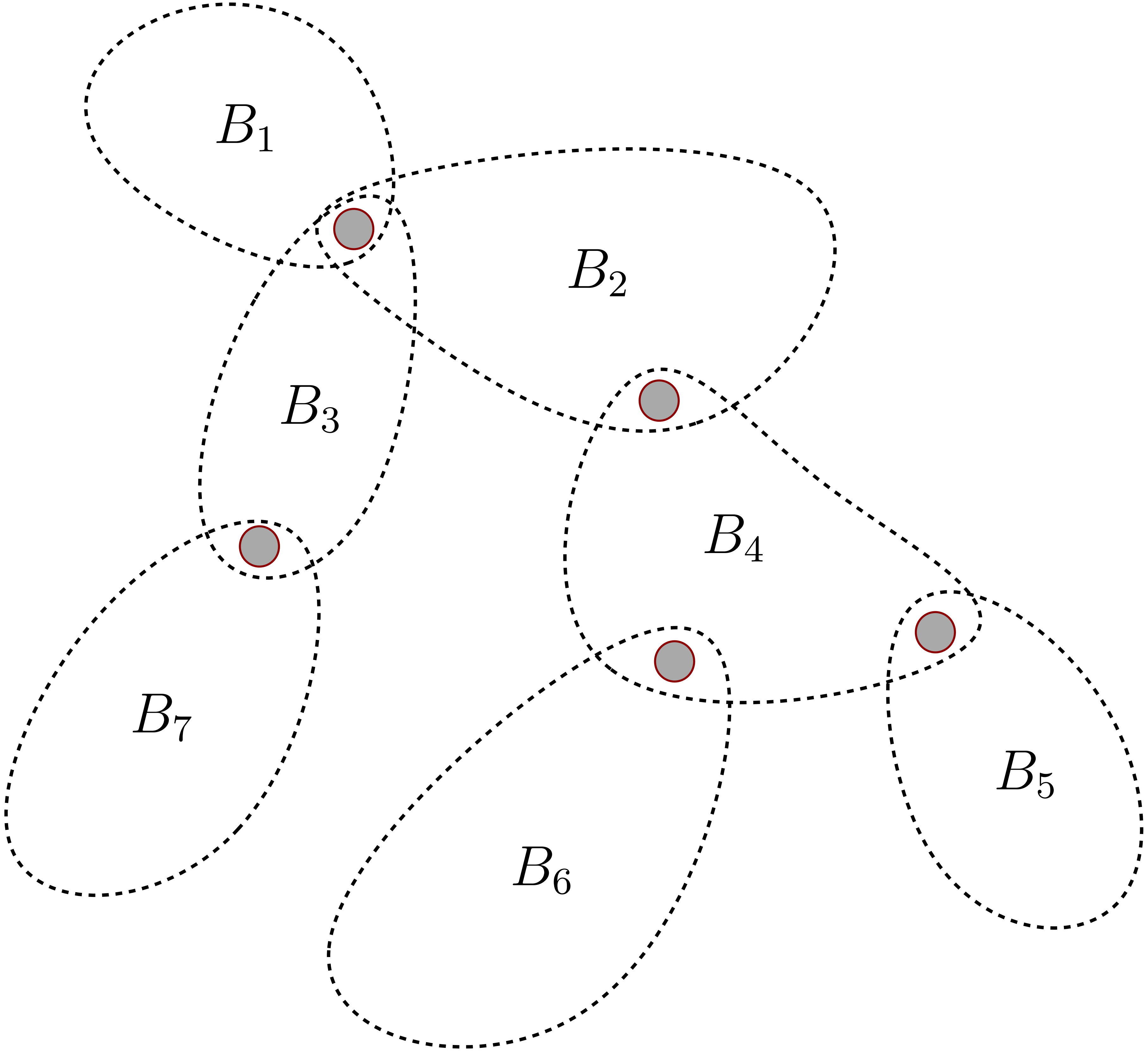
$[A < w < B]$ and $[C < w < D]$ are linear extensions of P' and P'' respectively

$[A < C < w < D < B]$ is a linear extension of $P' \cup P''$ is equal to respective "old" extensions when restricted to P' or P''.



$dim(B_i) \leq d$

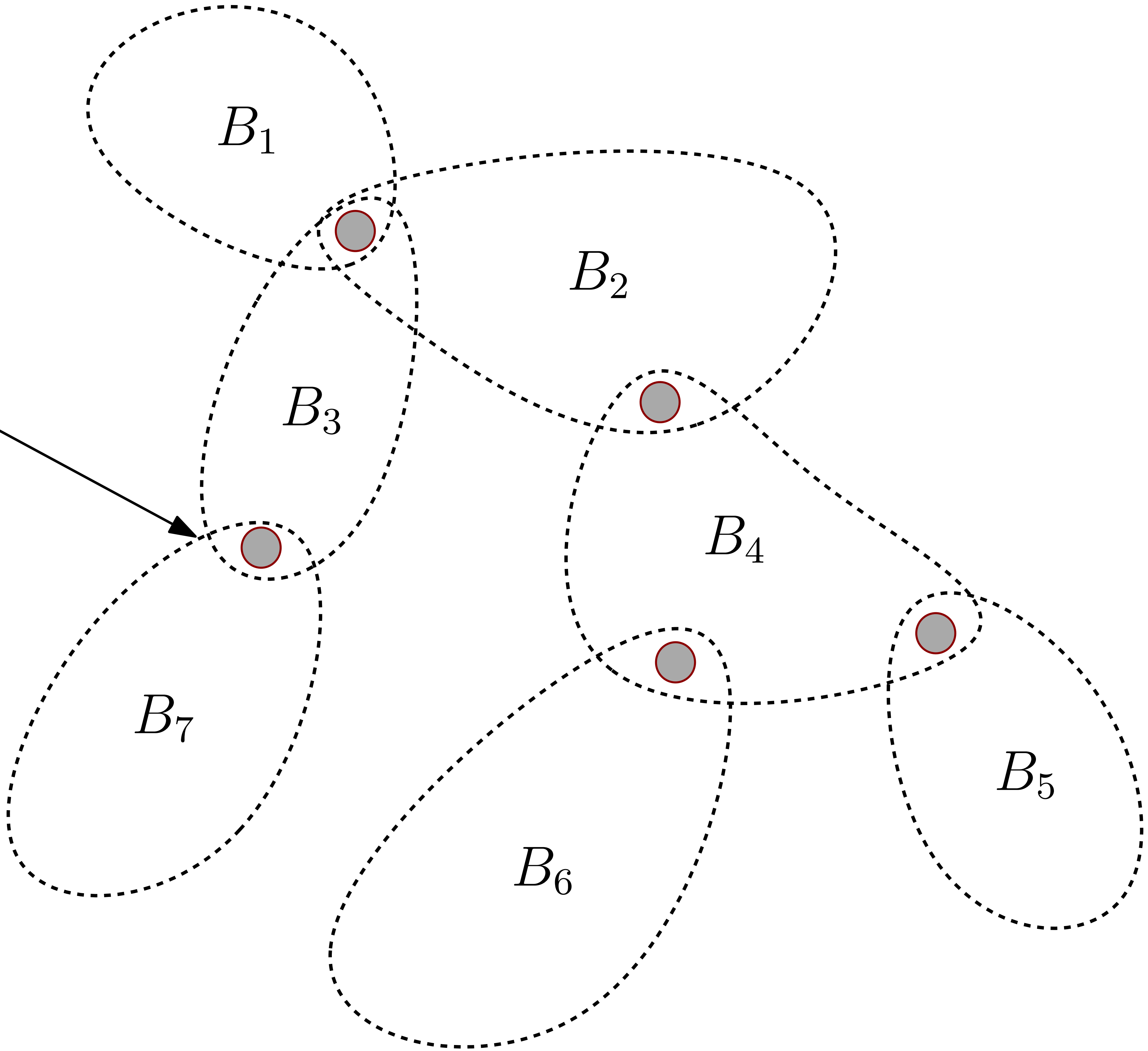
t - number of blocks



$dim(B_i) \leq d$

t - number of blocks

root of B_i - $\rho(B_i)$



$dim(B_i) \leq d$

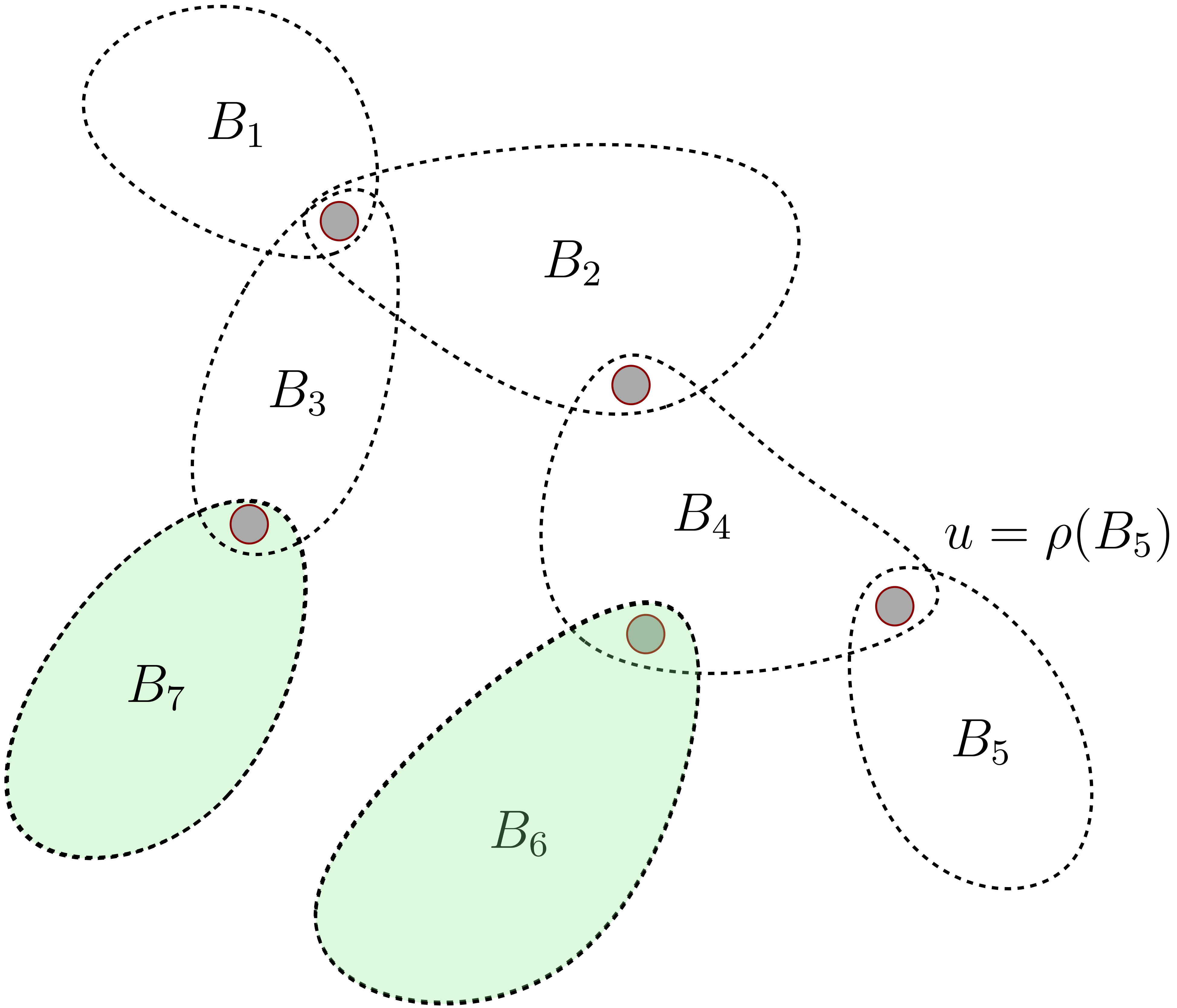
t - number of blocks

tail of u relative to B_i

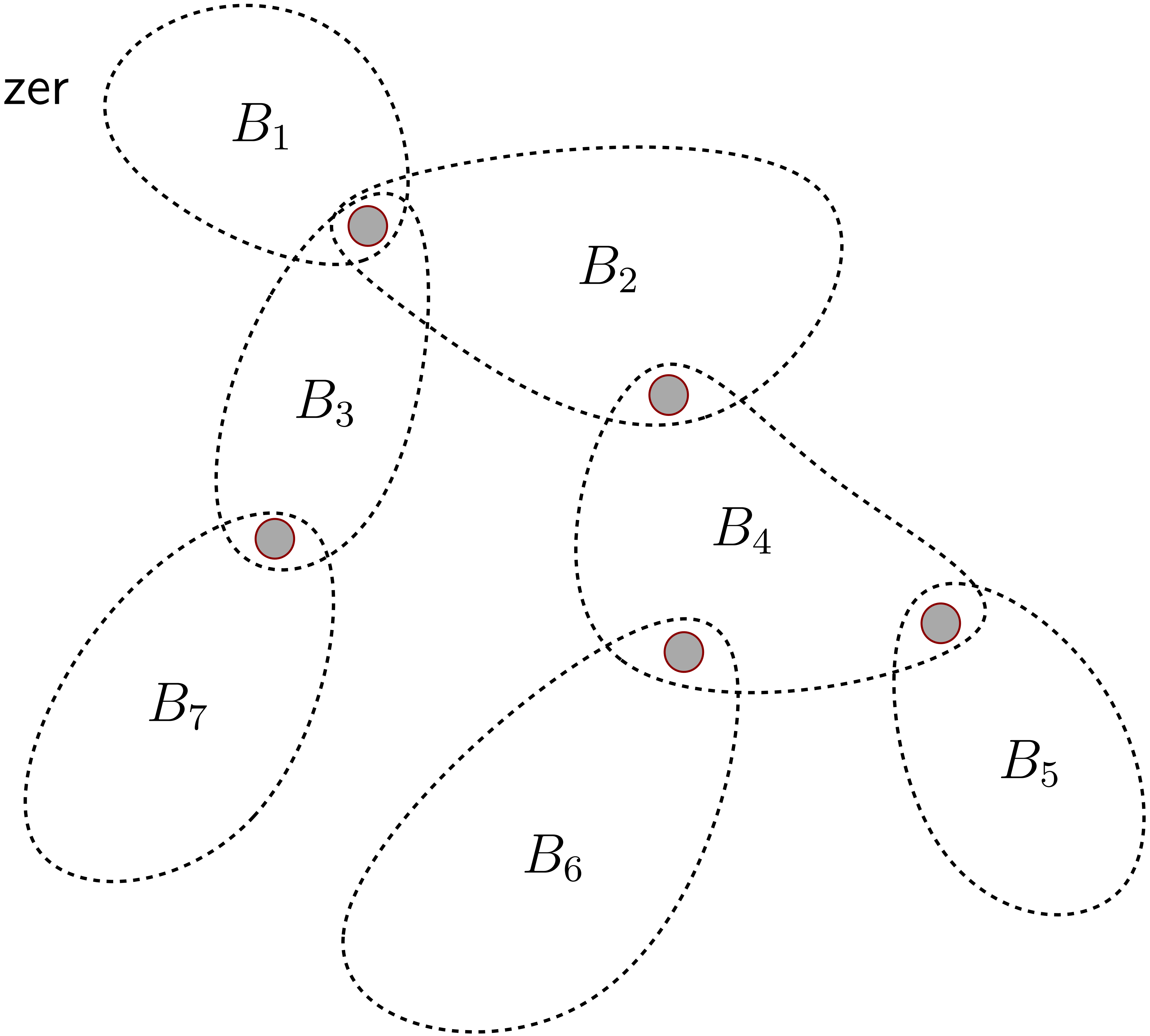
$T(u, B_i)$ for $u \in B_i$

$T(u, B_i) \subseteq \{u\} \cup B_{i+1} \cup \ldots \cup B_t$

$T(u, B_i)$ is the set of vertices from which you must go through u to reach B_i

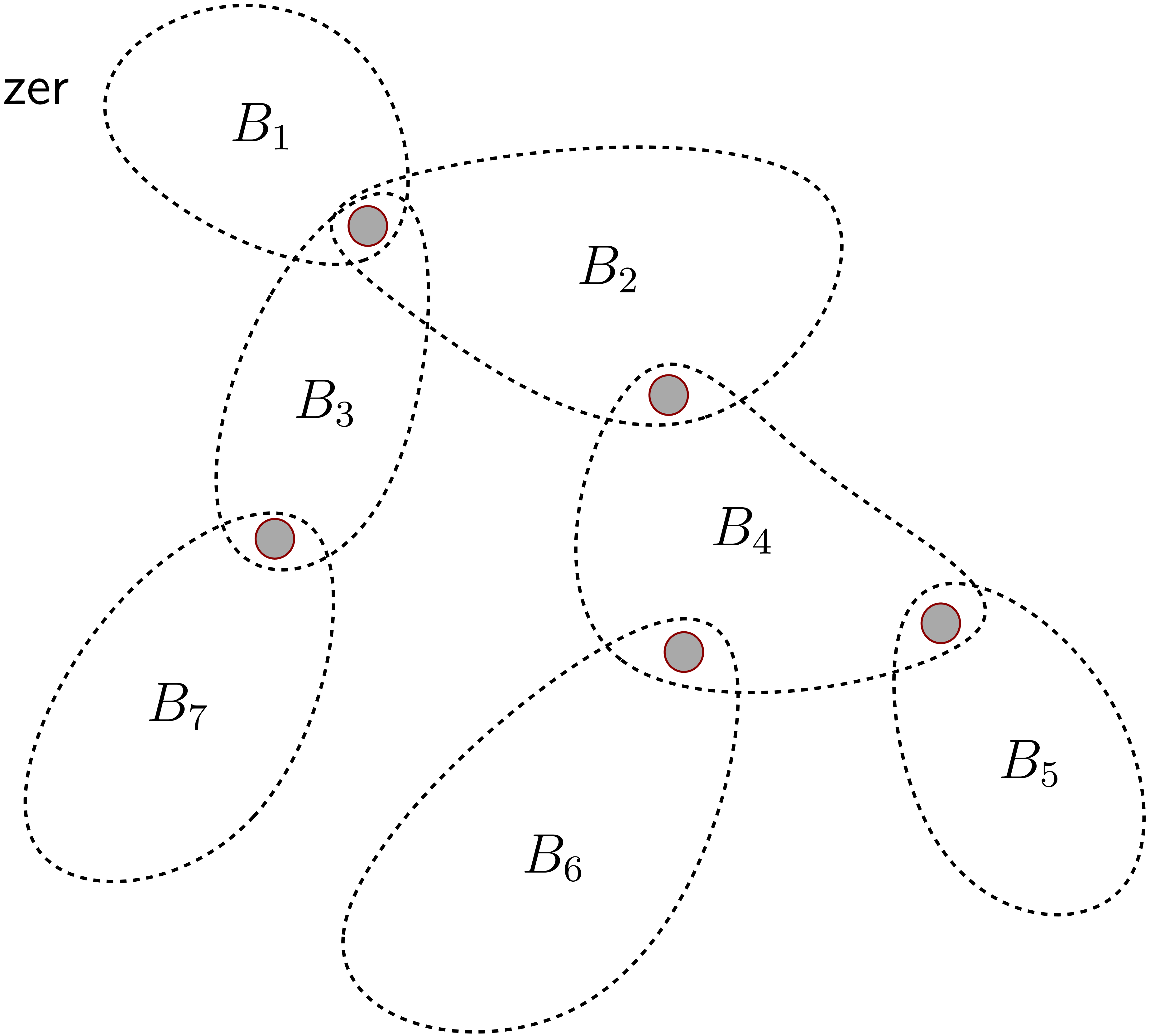


$L_1(B_i), L_2(B_i), \ldots, L_d(B_i)$ - realizer of B_i



$L_1(B_i), L_2(B_i), \ldots, L_d(B_i)$ - realizer of B_i

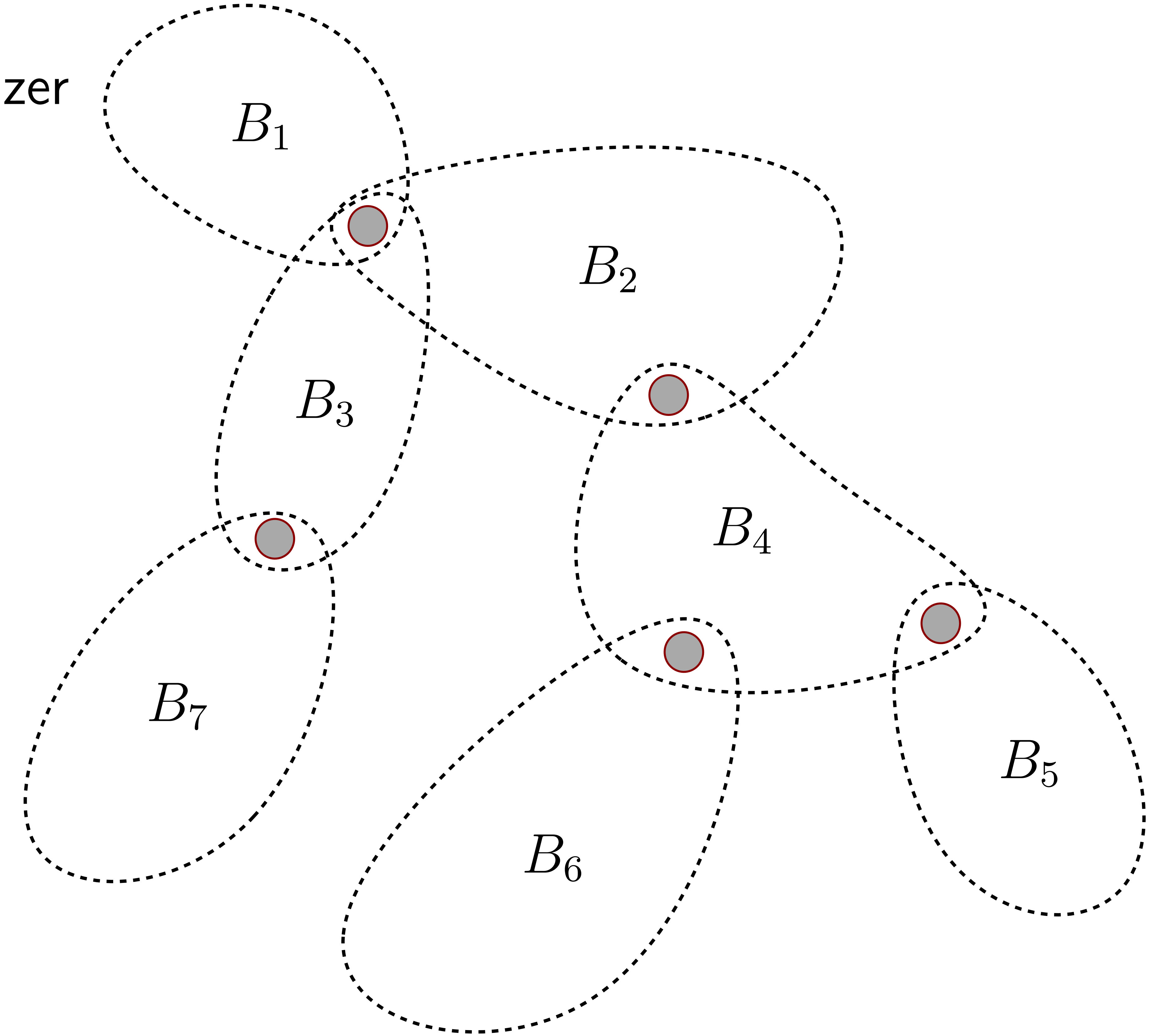
Fix j and take $L_j(B_1), L_j(B_2), \ldots, L_j(B_t)$



$L_1(B_i), L_2(B_i), \ldots, L_d(B_i)$ - realizer of B_i

Fix j and take $L_j(B_1), L_j(B_2), \ldots, L_j(B_t)$

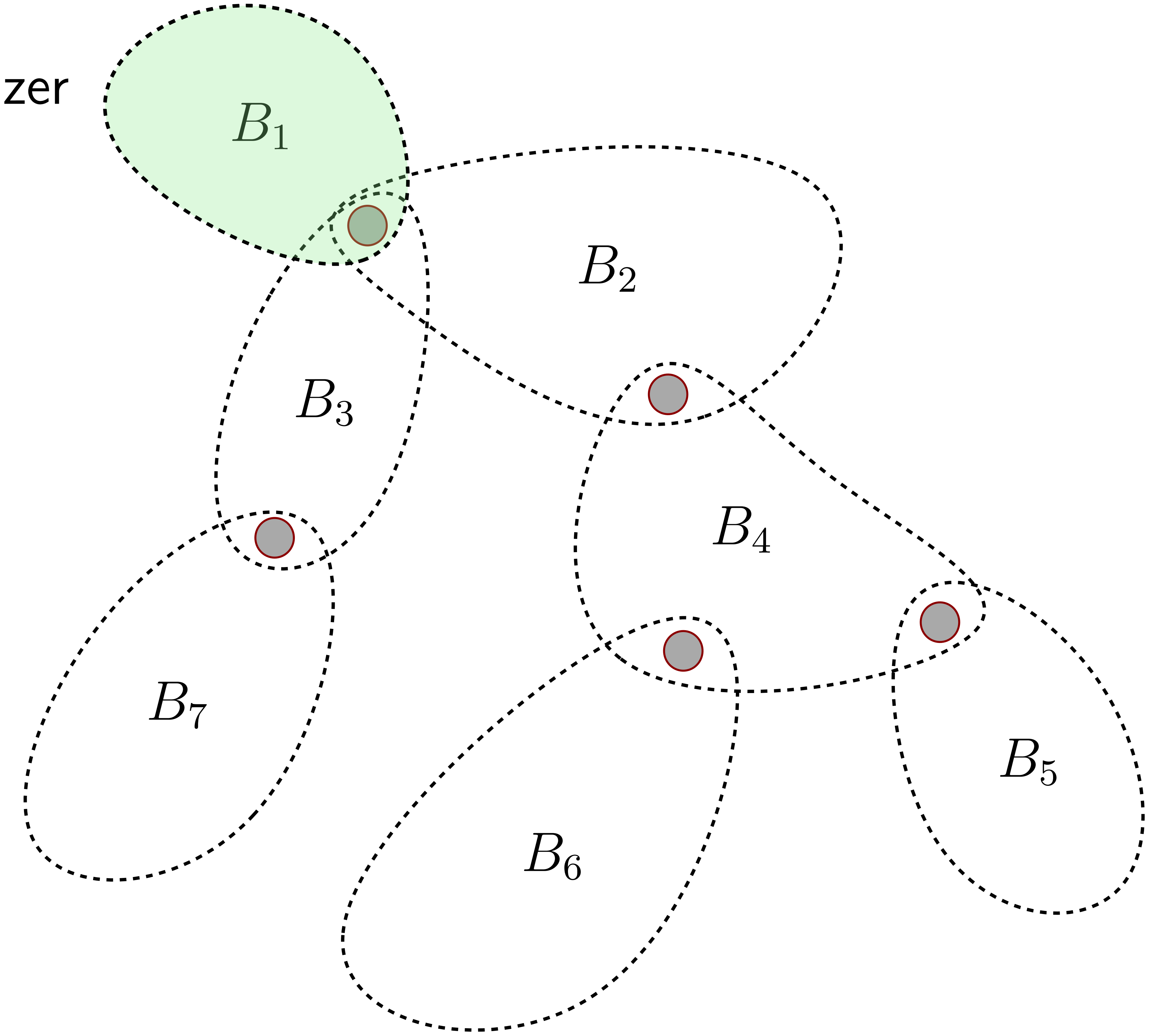
Iteratively construct linear extensions M_i of $P_i = B_1 \cup B_2 \cup \ldots \cup B_i$ using merge rule, starting from $M_1 = L_j(B_1)$



$L_1(B_i), L_2(B_i), \ldots, L_d(B_i)$ - realizer of B_i

Fix j and take $L_j(B_1), L_j(B_2), \ldots, L_j(B_t)$

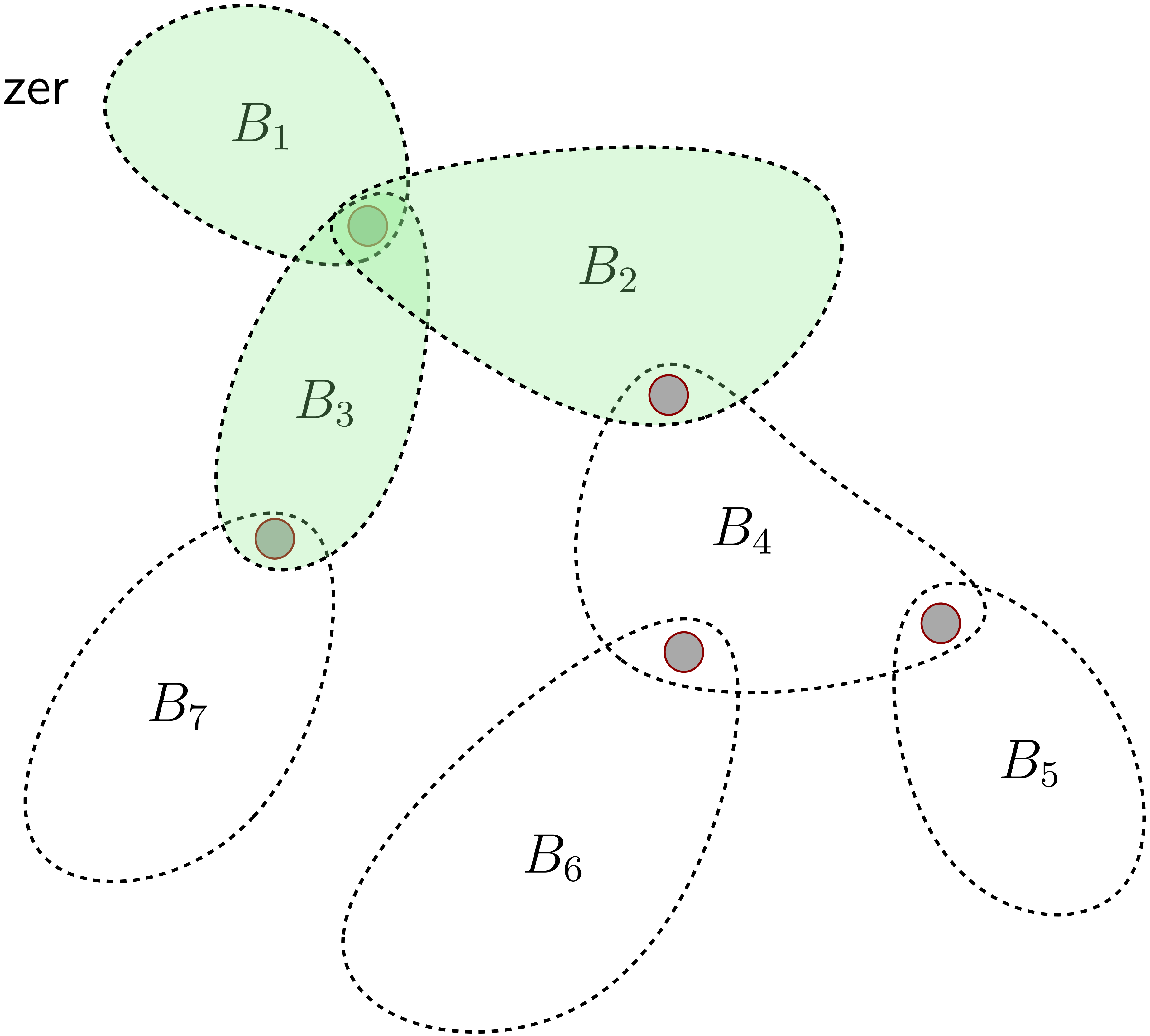
Iteratively construct linear extensions M_i of $P_i = B_1 \cup B_2 \cup \ldots \cup B_i$ using merge rule, starting from $M_1 = L_j(B_1)$



$L_1(B_i), L_2(B_i), \ldots, L_d(B_i)$ - realizer of B_i

Fix j and take $L_j(B_1), L_j(B_2), \ldots, L_j(B_t)$

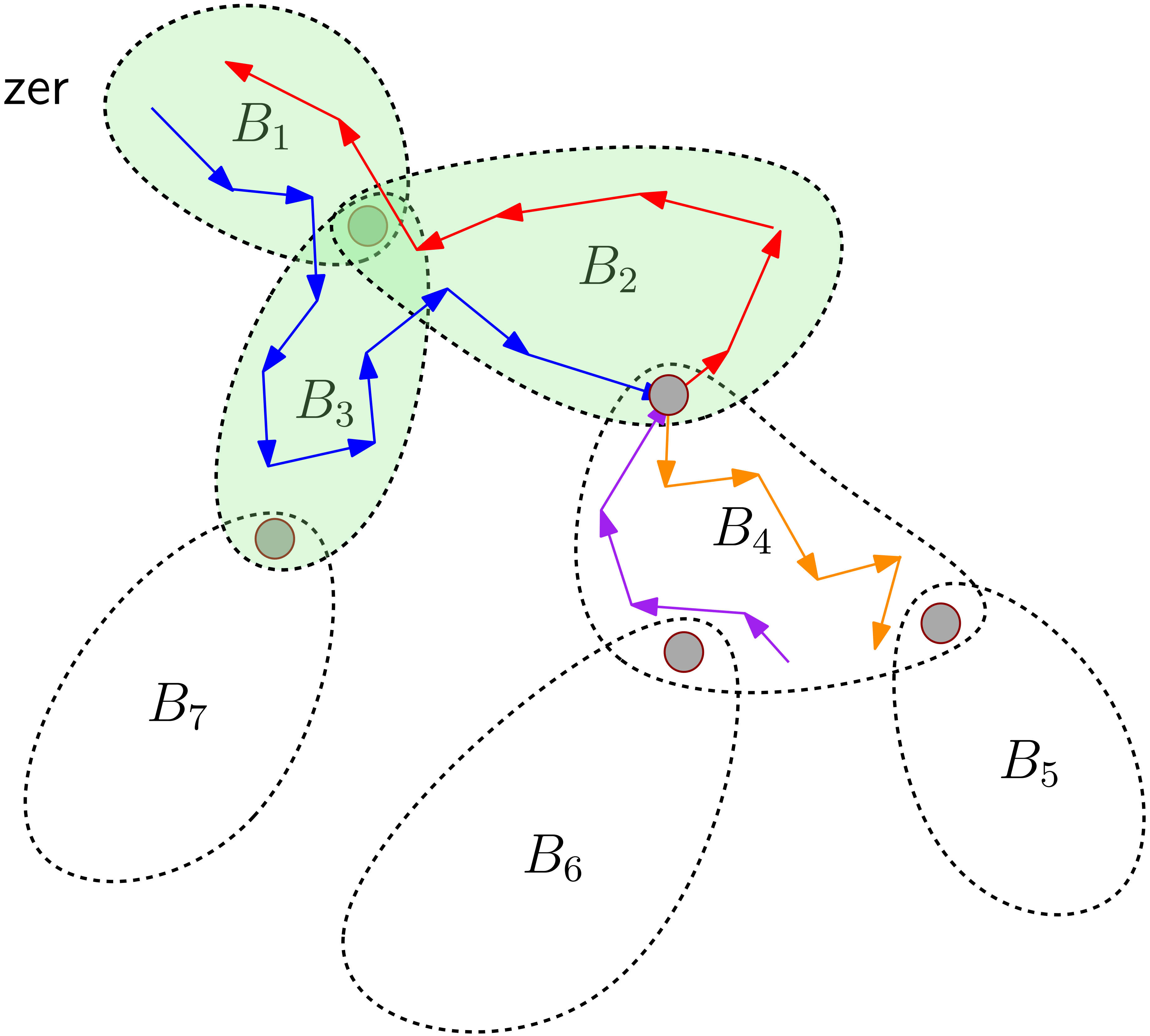
Iteratively construct linear extensions M_i of $P_i = B_1 \cup B_2 \cup \ldots \cup B_i$ using merge rule, starting from $M_1 = L_j(B_1)$



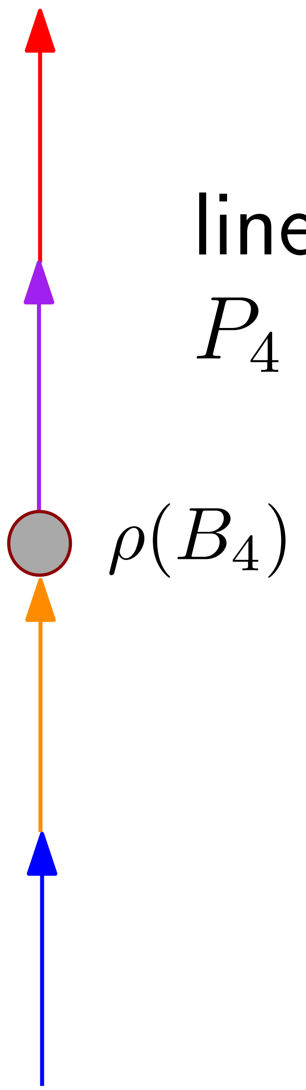
$L_1(B_i), L_2(B_i), \ldots, L_d(B_i)$ - realizer of B_i

Fix j and take $L_j(B_1), L_j(B_2), \ldots, L_j(B_t)$

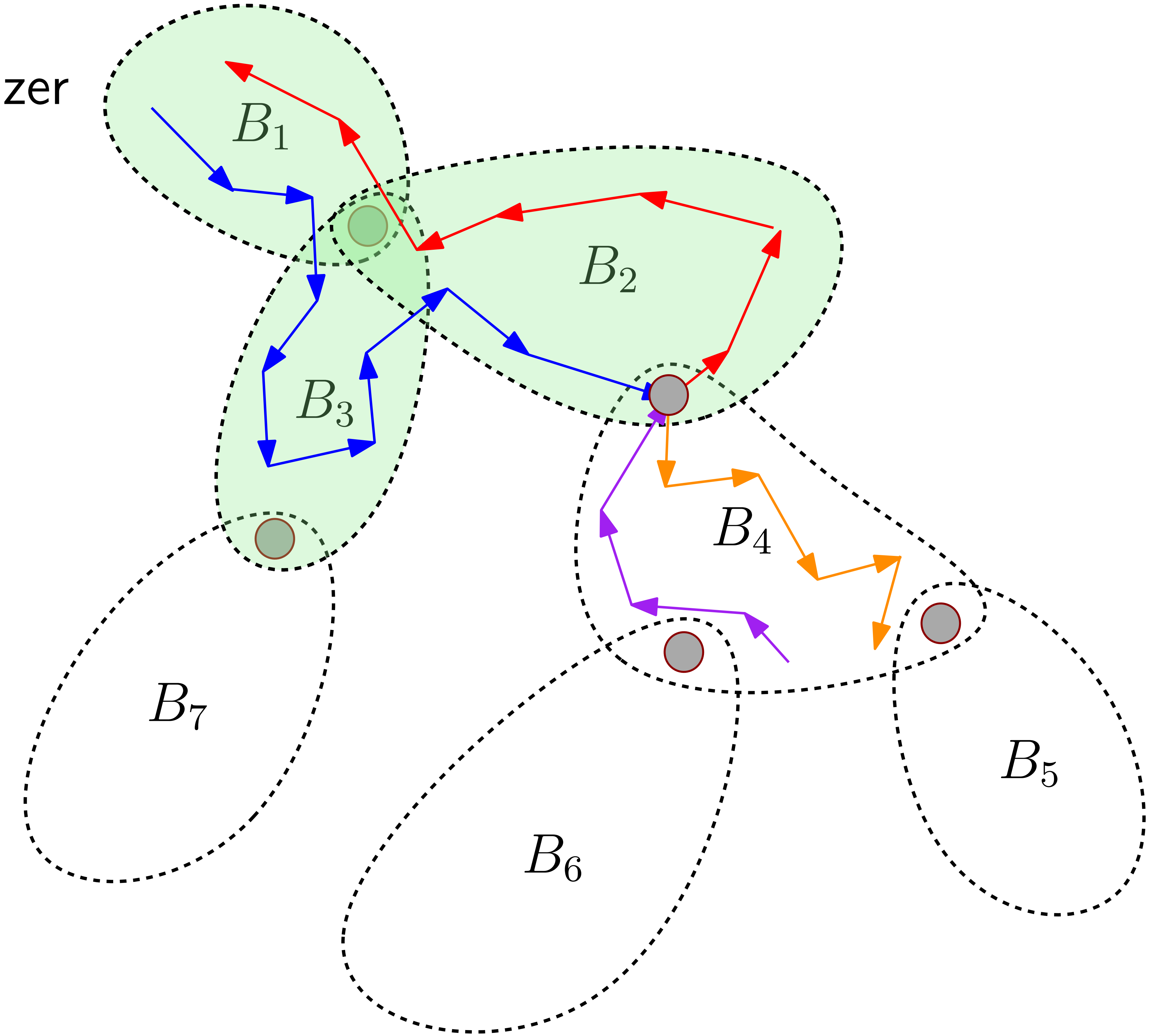
Iteratively construct linear extensions M_i of $P_i = B_1 \cup B_2 \cup \ldots \cup B_i$ using merge rule, starting from $M_1 = L_j(B_1)$



$L_1(B_i), L_2(B_i), \ldots, L_d(B_i)$ - realizer of B_i

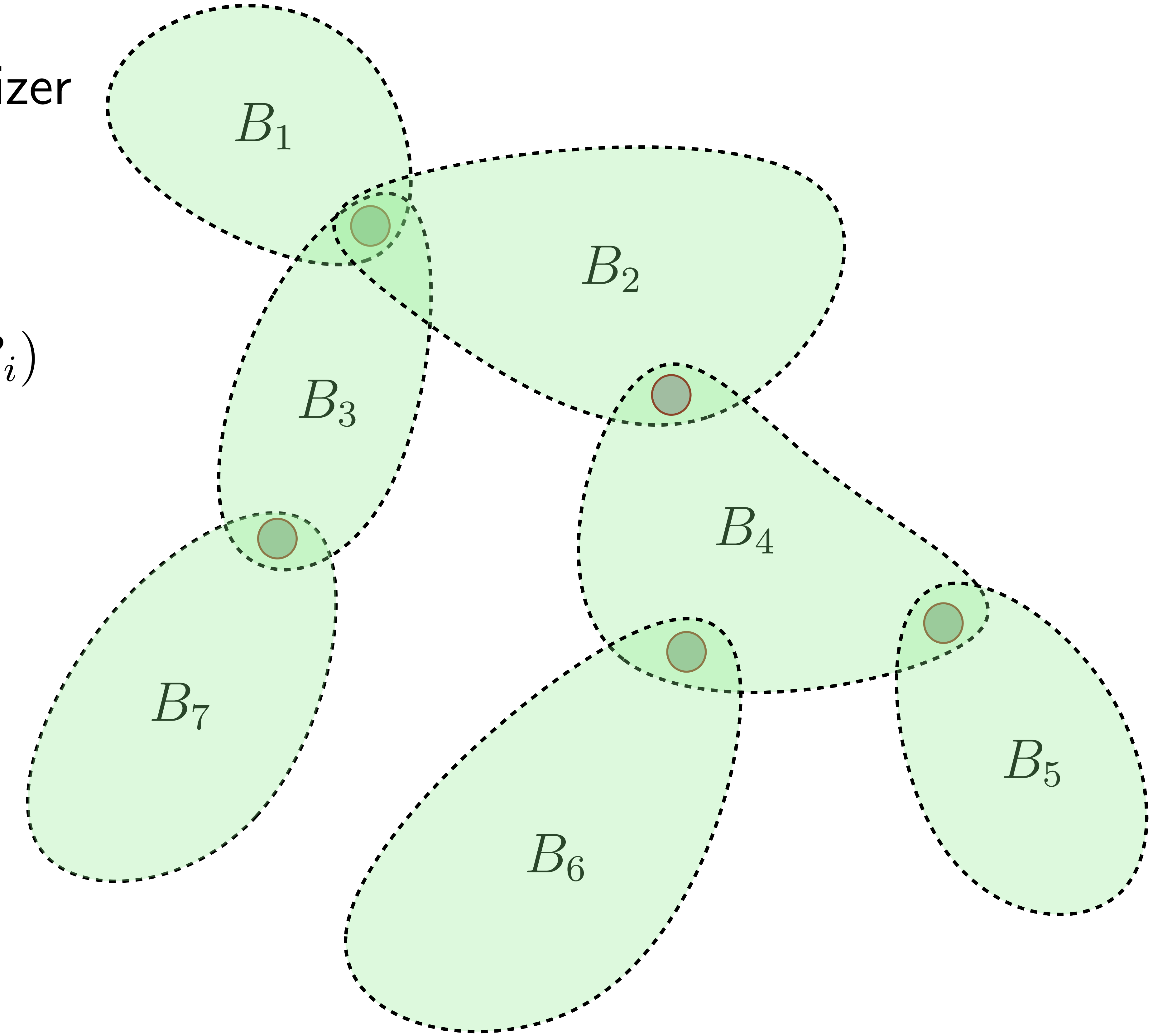


linear extension of
$P_4 = B_1 \cup \ldots \cup B_4$



$L_1(B_i), L_2(B_i), ..., L_d(B_i)$ - realizer of B_i

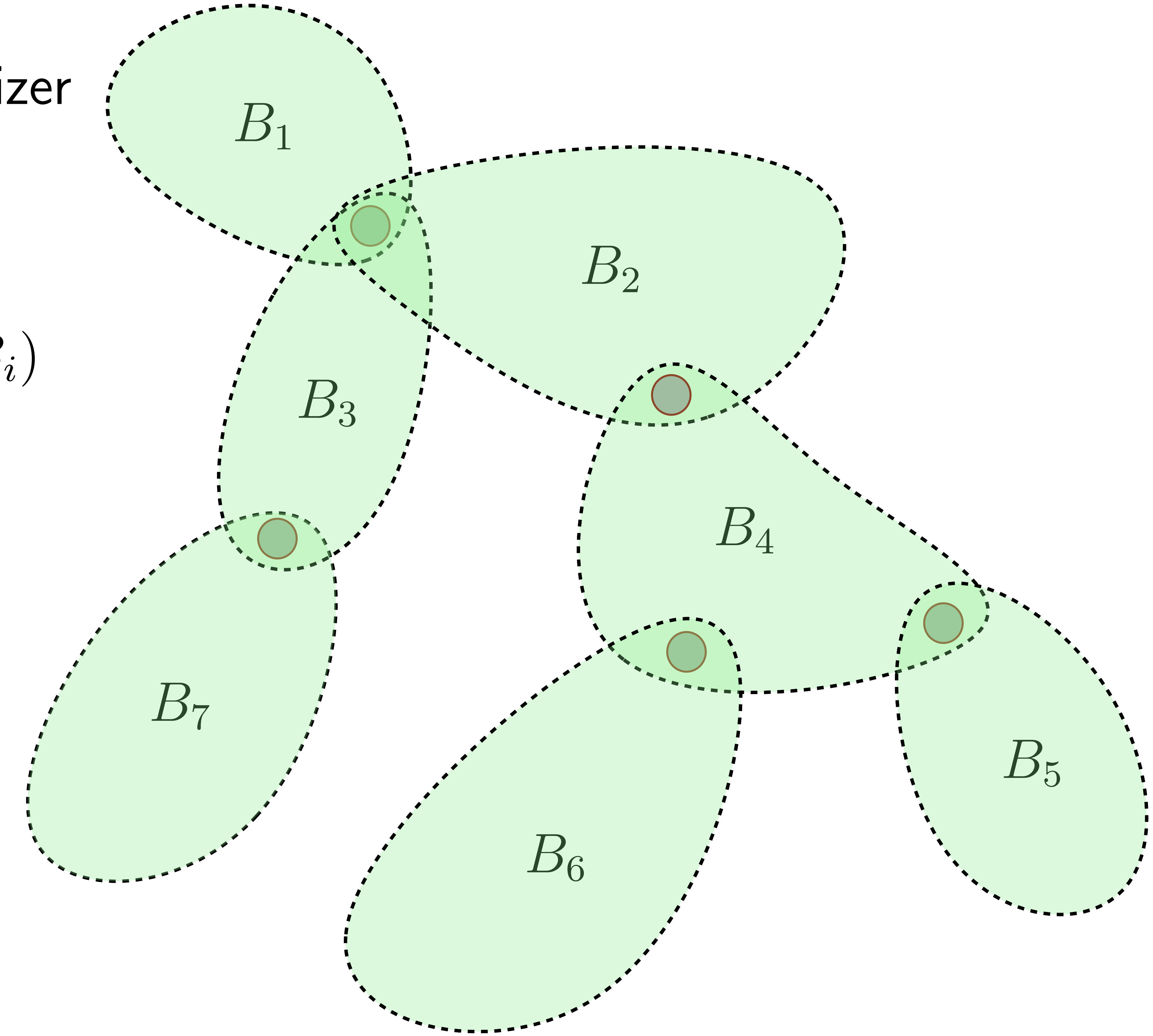
At the end we have L_j, a linear extension of P that equals $L_j(B_i)$ when restricted to B_i



$L_1(B_i), L_2(B_i), ..., L_d(B_i)$ - realizer of B_i

At the end we have L_j, a linear extension of P that equals $L_j(B_i)$ when restricted to B_i

This way we create $L_1, L_2, ..., L_d$, which is a realizer of P^*, an extension of P



Let $I \subseteq Inc(P)$ be the set of incomparable pairs in P that are not "killed" by $L_1, L_2, ..., L_d$.

Let $I \subseteq Inc(P)$ be the set of incomparable pairs in P that are not "killed" by $L_1, L_2, ..., L_d$.

Which means that $(x, y) \in I$ is still is $L_1 \cap L_2 \cap ... \cap L_d$ though we don't want it

Let $I \subseteq Inc(P)$ be the set of incomparable pairs in P that are not "killed" by $L_1, L_2, ..., L_d$.

Which means that $(x, y) \in I$ is still is $L_1 \cap L_2 \cap ... \cap L_d$ though we don't want it

if $I = \emptyset$, we are done. Now we show that if it is not empty, we can find two reversible sets $R_1, R_2 \subseteq I$ s.t. $R_1 \cup R_2 = I$

Let $I \subseteq Inc(P)$ be the set of incomparable pairs in P that are not "killed" by $L_1, L_2, ..., L_d$.

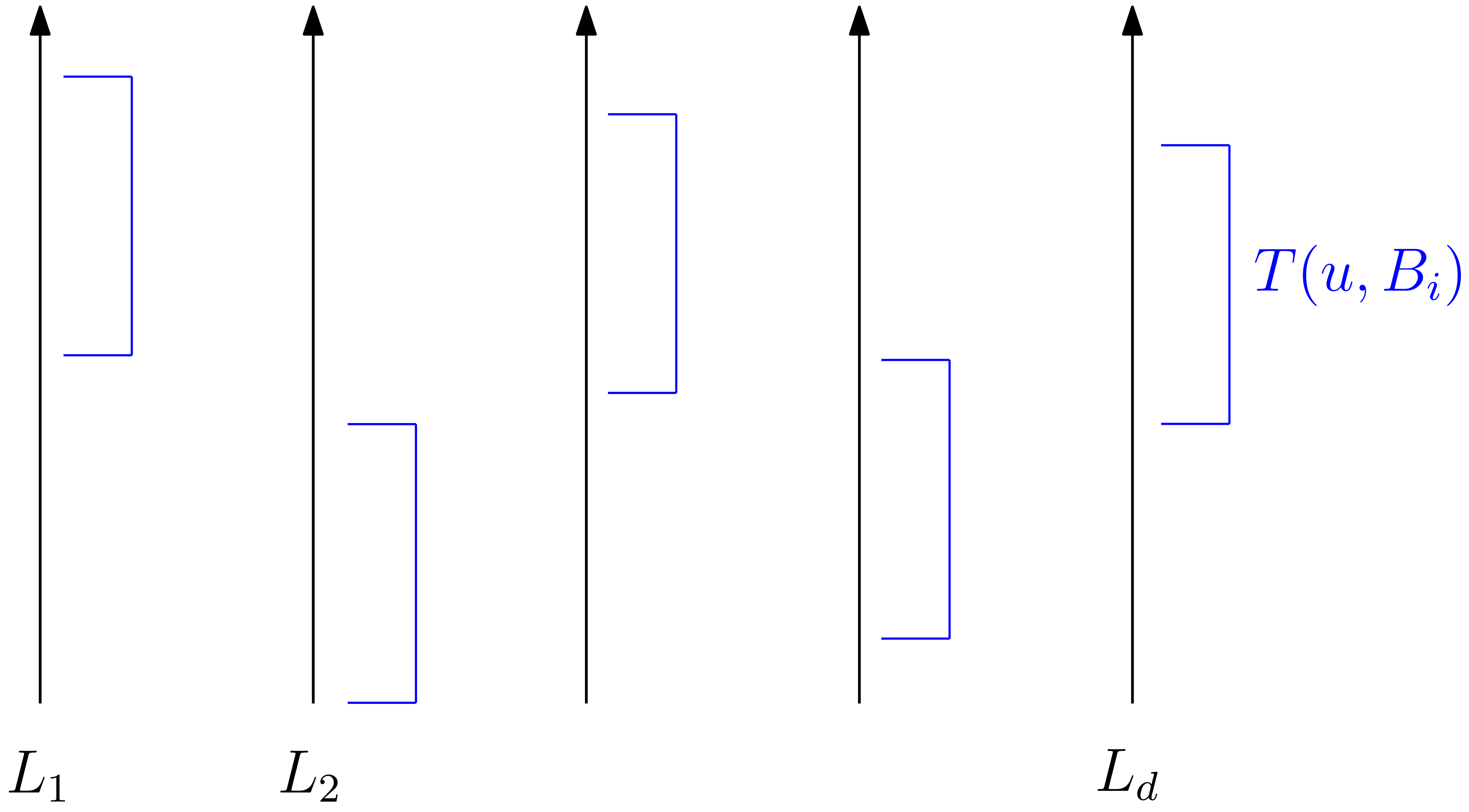
Which means that $(x, y) \in I$ is still is $L_1 \cap L_2 \cap ... \cap L_d$ though we don't want it

if $I = \emptyset$, we are done. Now we show that if it is not empty, we can find two reversible sets $R_1, R_2 \subseteq I$ s.t. $R_1 \cup R_2 = I$

This will end the proof, because we will be able to add two linear extensions of P - L_{d+1} and L_{d+2} s.t. $L_1, L_2, ..., L_d, L_{d+1}, L_{d+2}$ will be a realizer of P

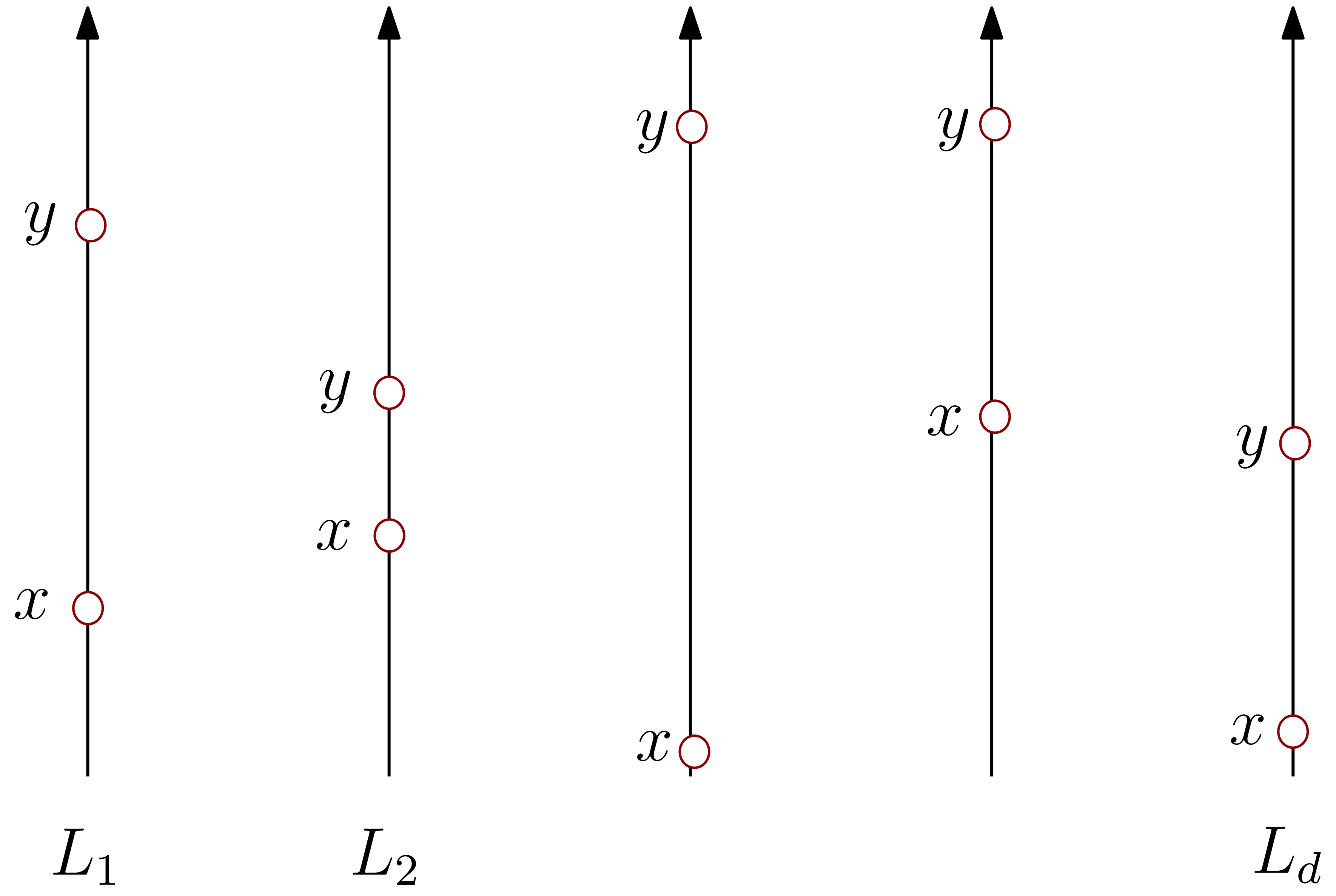
Interval property for tails. Tails form intervals in L_j for all $1 \leq j \leq d$

Interval property for tails. Tails form intervals in L_j for all $1 \leq j \leq d$

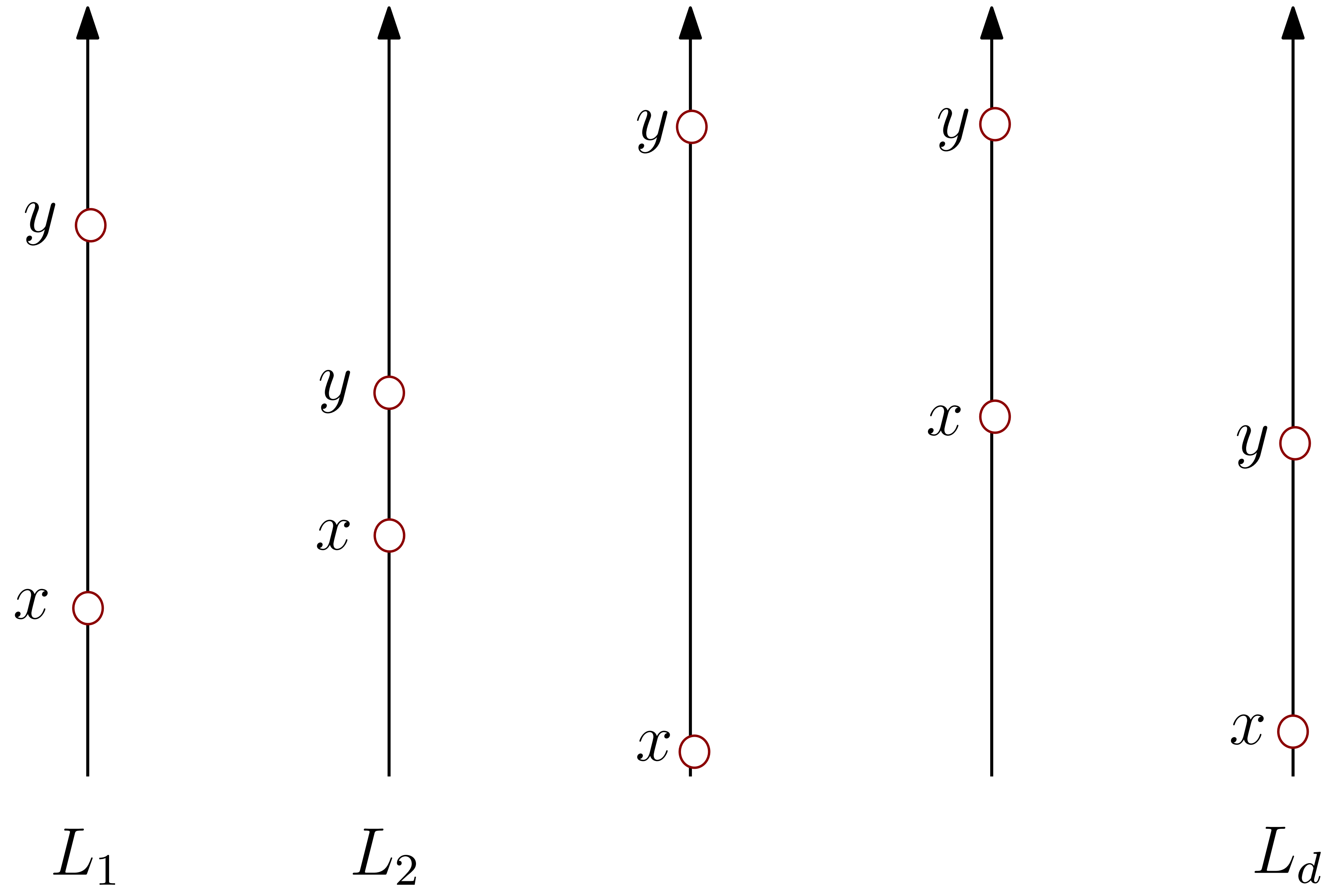


Take any $(x, y) \in I$

Take any $(x, y) \in I$



Take any $(x, y) \in I$



x and y cannot belong to one block

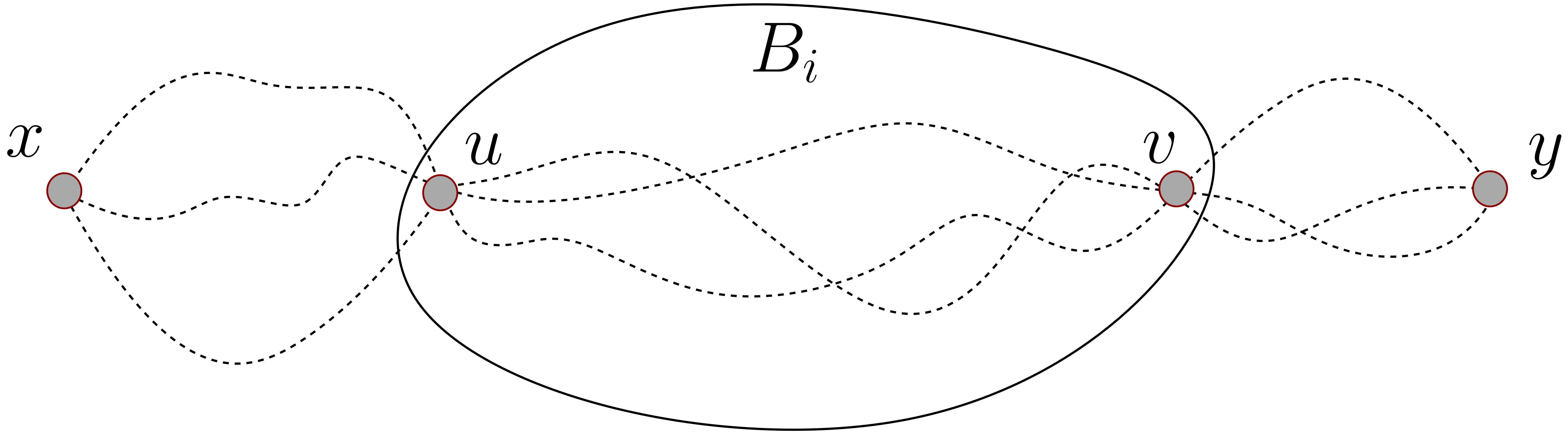
Consider all paths from x to y in cover graph. Let i be the smallest index s.t. every path from x to y passes through B_i

Consider all paths from x to y in cover graph. Let i be the smallest index s.t. every path from x to y passes through B_i

Let u and v be articulation points through which these paths enter and leave B_i

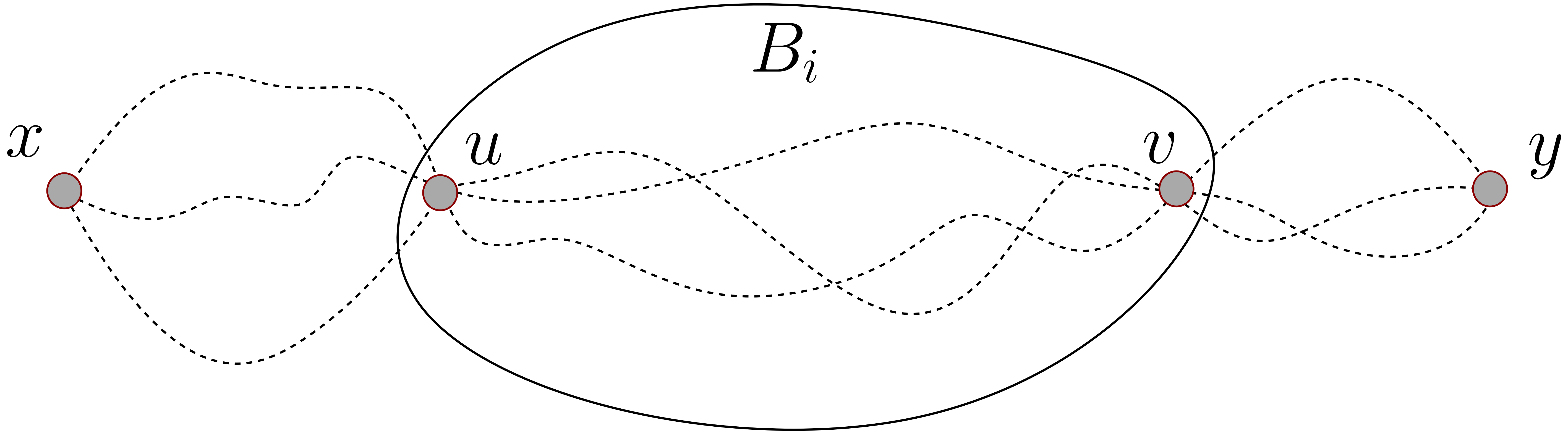
Consider all paths from x to y in cover graph. Let i be the smallest index s.t. every path from x to y passes through B_i

Let u and v be articulation points through which these paths enter and leave B_i



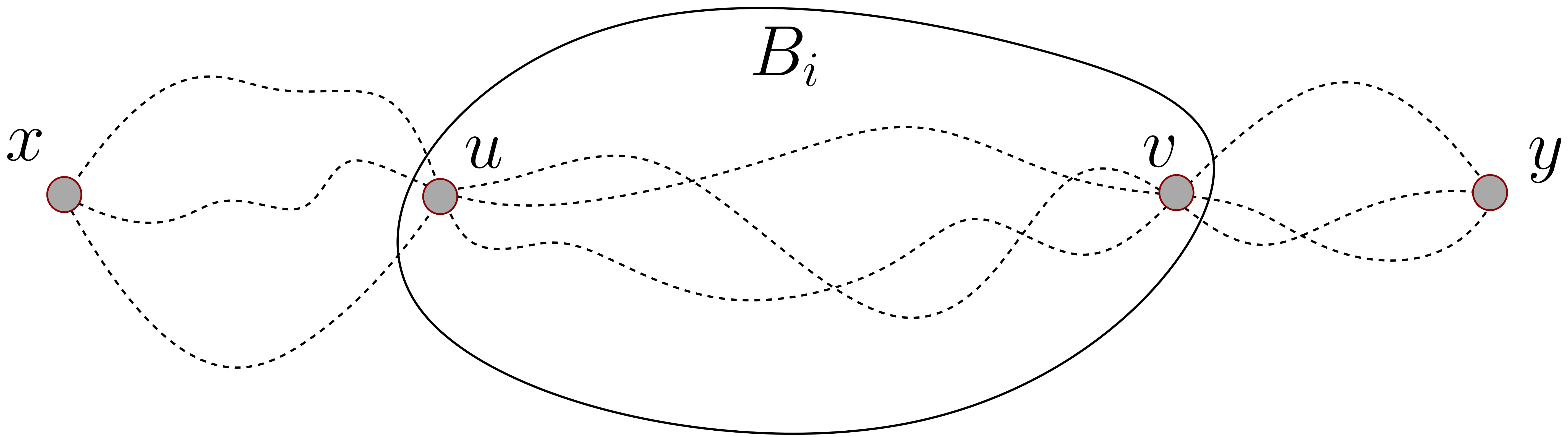
Consider all paths from x to y in cover graph. Let i be the smallest index s.t. every path from x to y passes through B_i

Let u and v be articulation points through which these paths enter and leave B_i



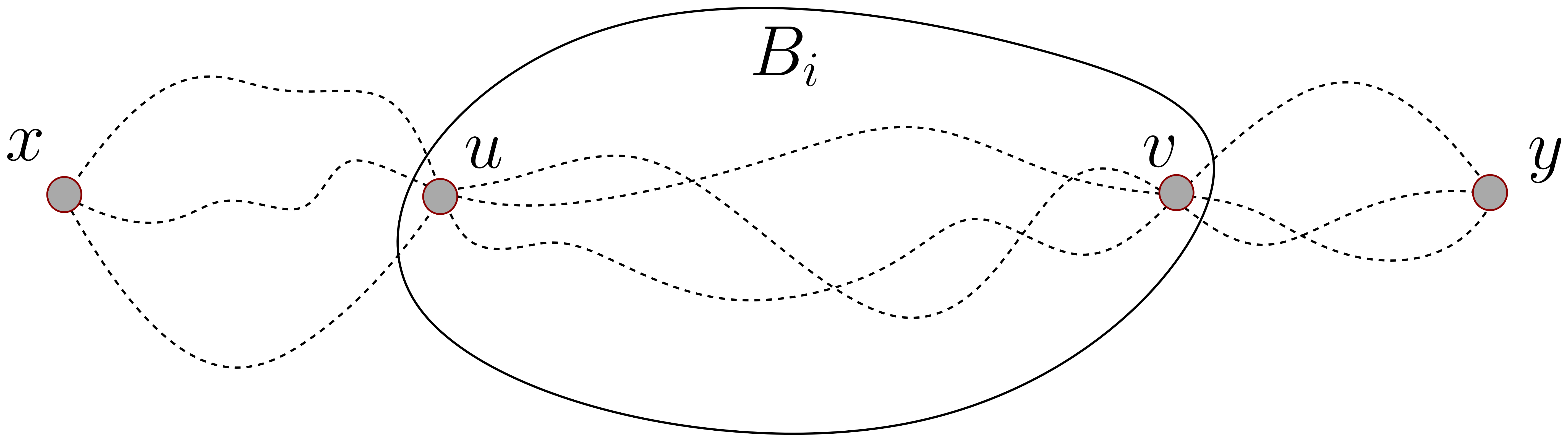
Note: It can happen that $x = u$ or $y = v$, incoming arguments still hold in these cases.

Claim. $x \in T(u, B_i)$, $y \notin T(u, B_i)$, $y \in T(v, B_i)$, $x \notin T(v, B_i)$



Claim. $x \in T(u, B_i)$, $y \notin T(u, B_i)$, $y \in T(v, B_i)$, $x \notin T(v, B_i)$

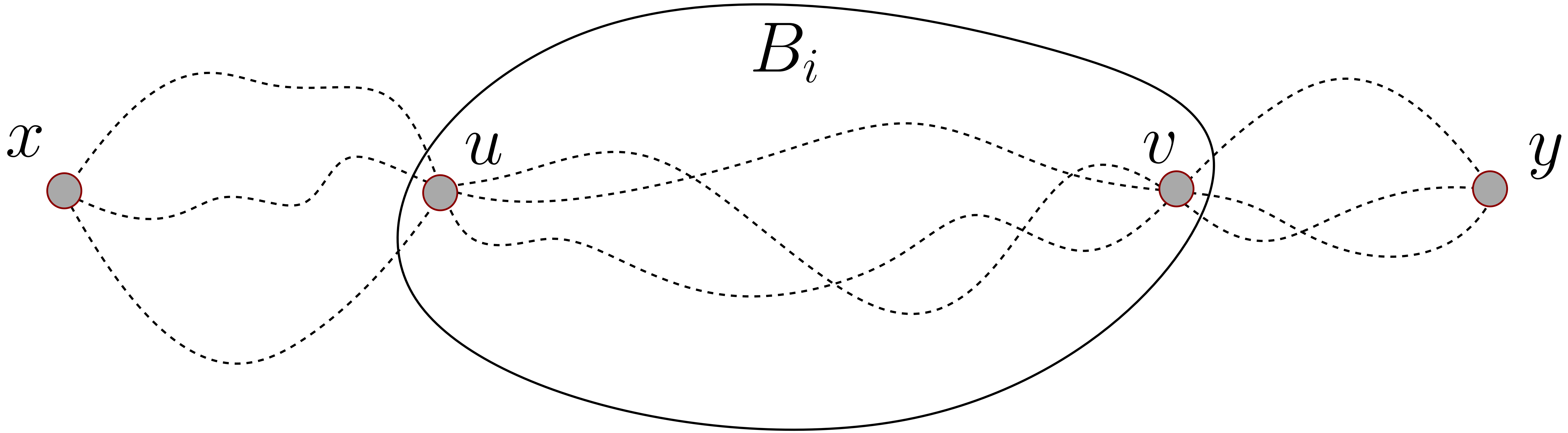
Claim. $T(u, B_i) \cap T(v, B_i) = \emptyset$



Claim. $x \in T(u, B_i)$, $y \notin T(u, B_i)$, $y \in T(v, B_i)$, $x \notin T(v, B_i)$

Claim. $T(u, B_i) \cap T(v, B_i) = \emptyset$

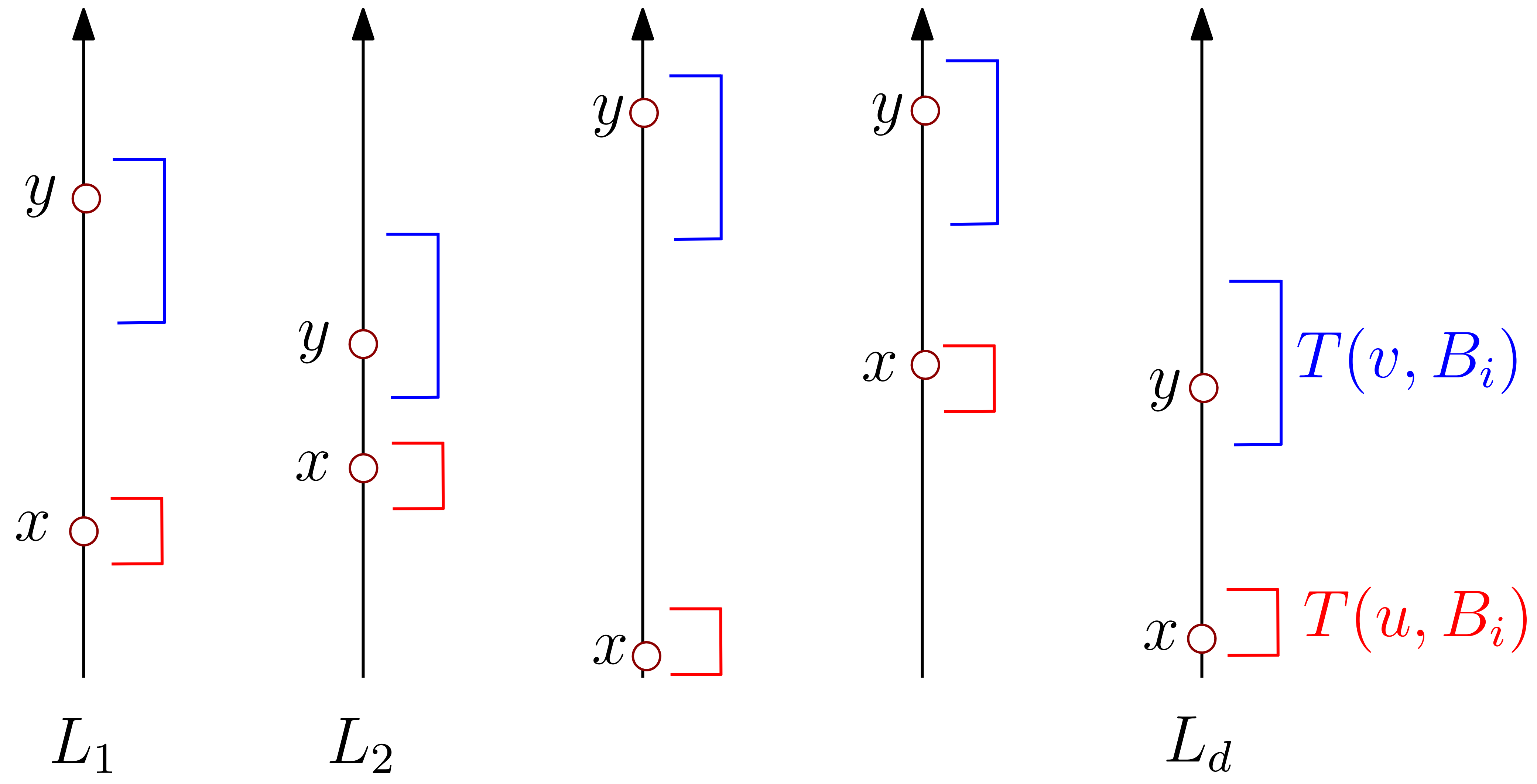
Claim. $u < v$ in P



Claim. $x \in T(u, B_i)$, $y \notin T(u, B_i)$, $y \in T(v, B_i)$, $x \notin T(v, B_i)$

Claim. $T(u, B_i) \cap T(v, B_i) = \emptyset$

Claim. $u < v$ in P



Claim. At least one of the two following statements hold:

(1) for all $y' \geq x$ in P, $y' \in T(u, B_i)$ (this implies $y' < y$ in P^*)

(2) for all $x' \leq y$ in P, $x' \in T(v, B_i)$ (this implies $x < x'$ in P^*)

Claim. At least one of the two following statements hold:

(1) for all $y' \geq x$ in P, $y' \in T(u, B_i)$ (this implies $y' < y$ in P^*)

(2) for all $x' \leq y$ in P, $x' \in T(v, B_i)$ (this implies $x < x'$ in P^*)

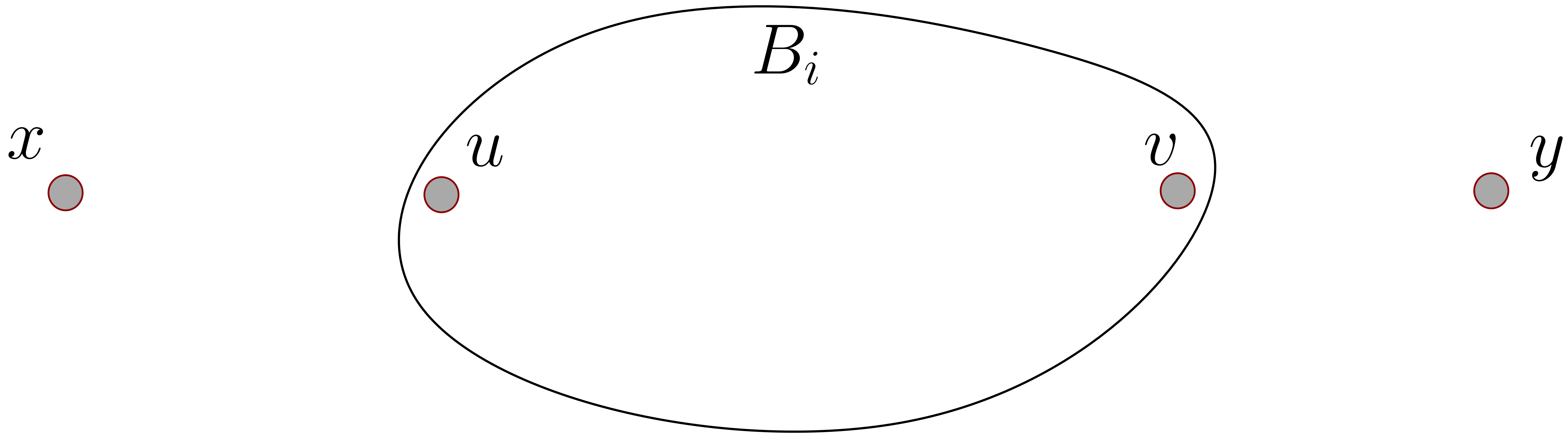
If not:

Claim. At least one of the two following statements hold:

(1) for all $y' \geq x$ in P, $y' \in T(u, B_i)$ (this implies $y' < y$ in P^*)

(2) for all $x' \leq y$ in P, $x' \in T(v, B_i)$ (this implies $x < x'$ in P^*)

If not:

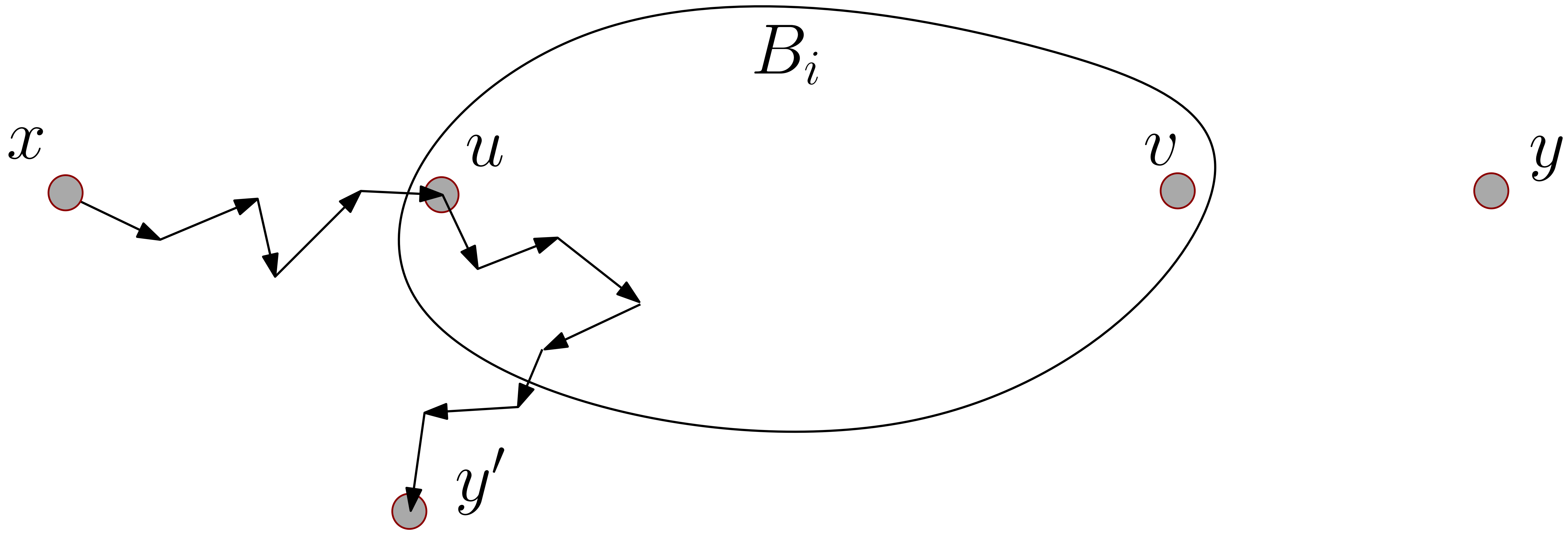


Claim. At least one of the two following statements hold:

(1) for all $y' \geq x$ in P, $y' \in T(u, B_i)$ (this implies $y' < y$ in P^*)

(2) for all $x' \leq y$ in P, $x' \in T(v, B_i)$ (this implies $x < x'$ in P^*)

If not:

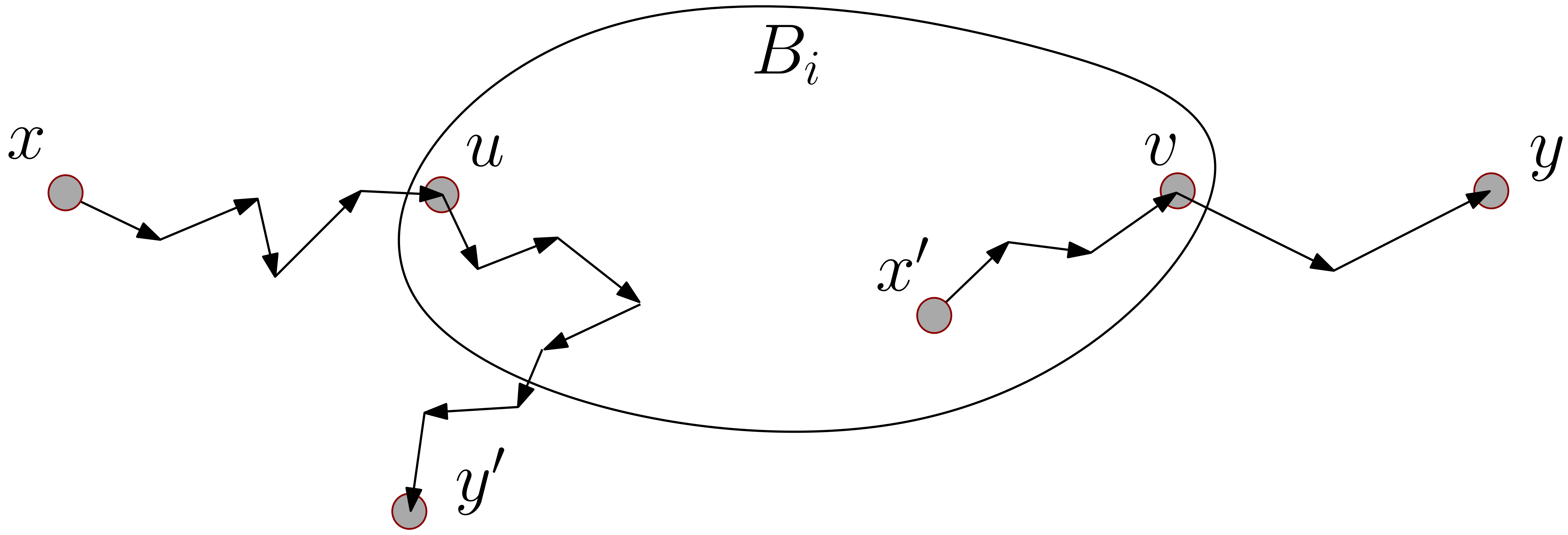


Claim. At least one of the two following statements hold:

(1) for all $y' \geq x$ in P, $y' \in T(u, B_i)$ (this implies $y' < y$ in P^*)

(2) for all $x' \leq y$ in P, $x' \in T(v, B_i)$ (this implies $x < x'$ in P^*)

If not:

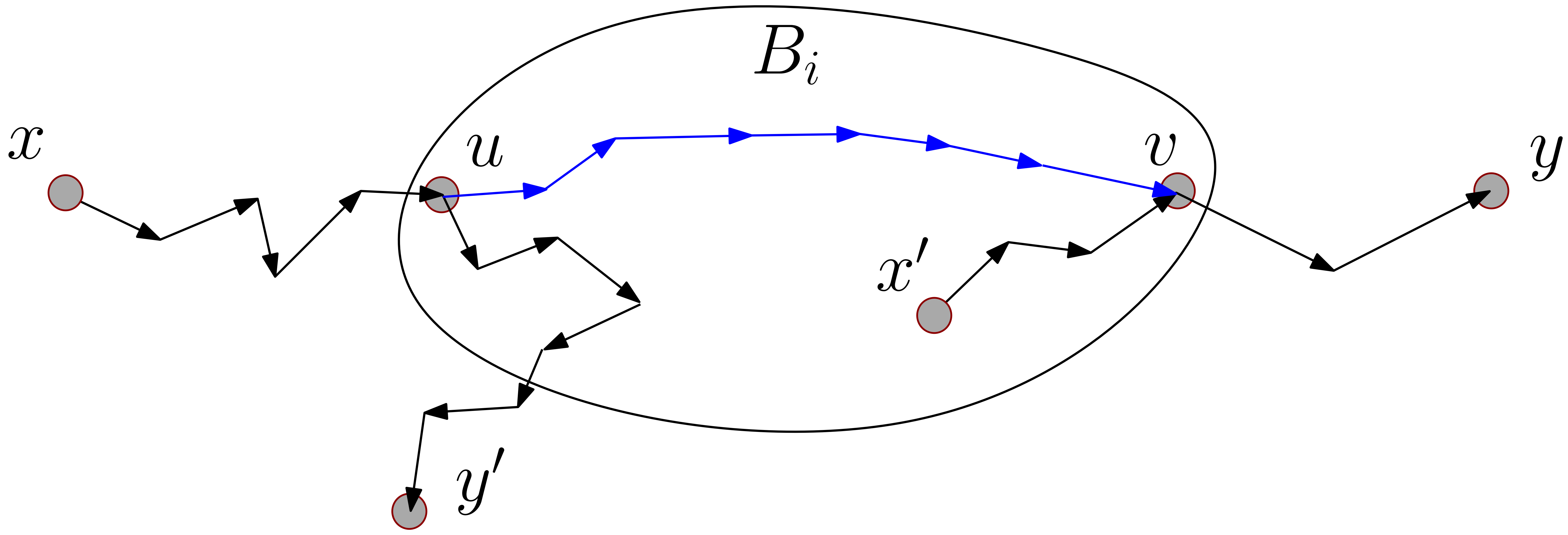


Claim. At least one of the two following statements hold:

(1) for all $y' \geq x$ in P, $y' \in T(u, B_i)$ (this implies $y' < y$ in P^*)

(2) for all $x' \leq y$ in P, $x' \in T(v, B_i)$ (this implies $x < x'$ in P^*)

If not:

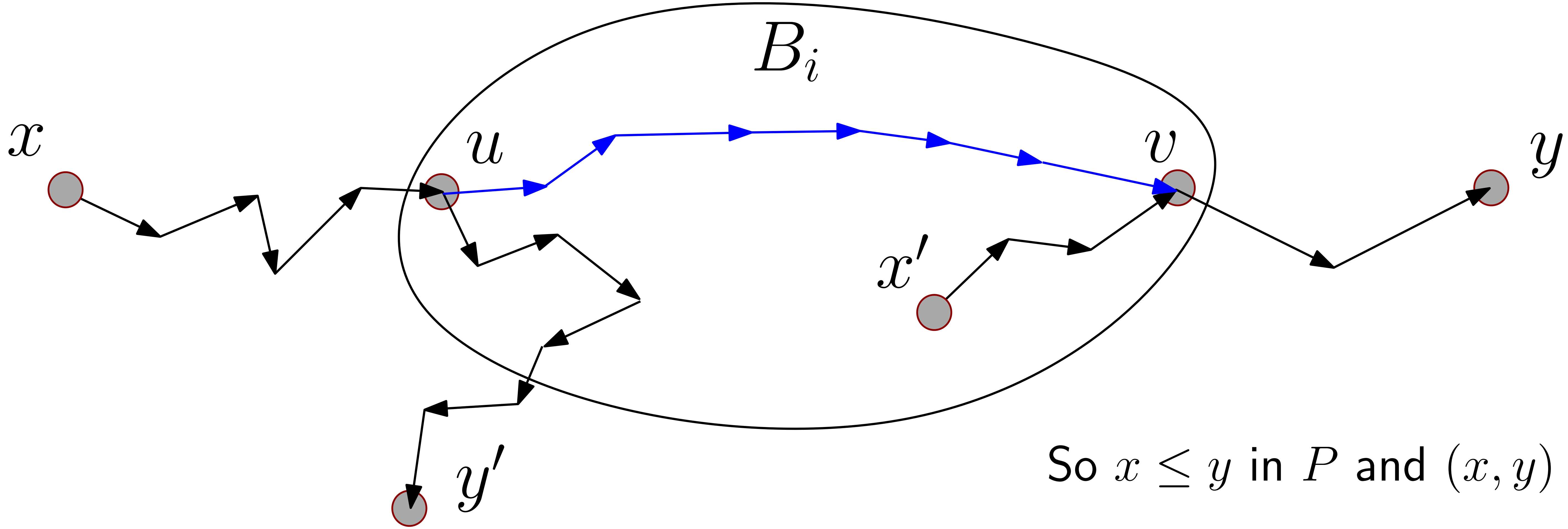


Claim. At least one of the two following statements hold:

(1) for all $y' \geq x$ in P, $y' \in T(u, B_i)$ (this implies $y' < y$ in P^*)

(2) for all $x' \leq y$ in P, $x' \in T(v, B_i)$ (this implies $x < x'$ in P^*)

If not:



So $x \leq y$ in P and $(x, y) \notin I$

Claim. At least one of the two following statements hold:

(1) for all $y' \geq x$ in P, $y' \in T(u, B_i)$ (this implies $y' < y$ in P^*) R_1

(2) for all $x' \leq y$ in P, $x' \in T(v, B_i)$ (this implies $x < x'$ in P^*) R_2

Claim. At least one of the two following statements hold:

(1) for all $y' \geq x$ in P, $y' \in T(u, B_i)$ (this implies $y' < y$ in P^*) $\quad R_1$

(2) for all $x' \leq y$ in P, $x' \in T(v, B_i)$ (this implies $x < x'$ in P^*) $\quad R_2$

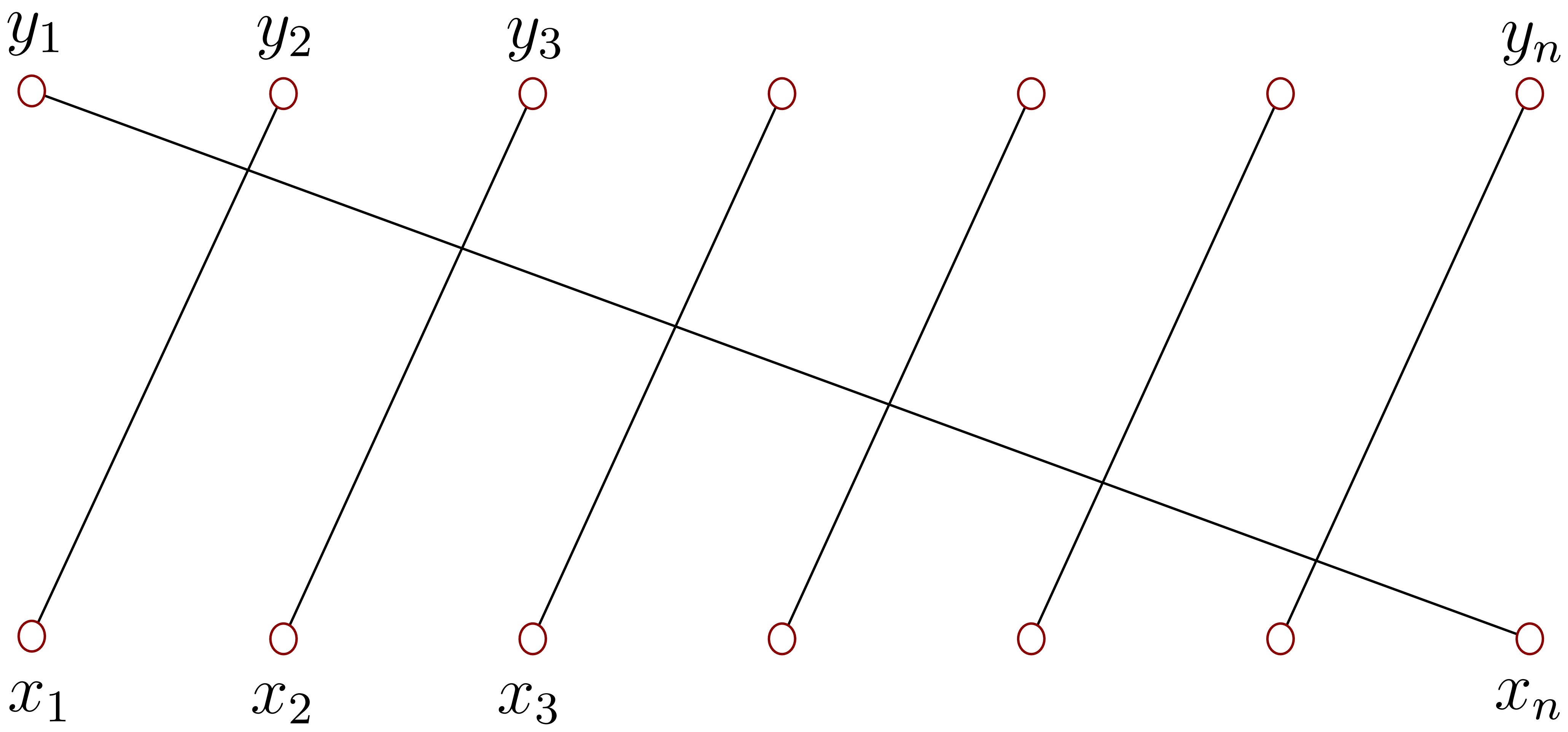
Now we prove that R_1 is reversible. Assume it is not, then it has an alternating cycle:

Claim. At least one of the two following statements hold:

(1) for all $y' \geq x$ in P, $y' \in T(u, B_i)$ (this implies $y' < y$ in P^*) R_1

(2) for all $x' \leq y$ in P, $x' \in T(v, B_i)$ (this implies $x < x'$ in P^*) R_2

Now we prove that R_1 is reversible. Assume it is not, then it has an alternating cycle:

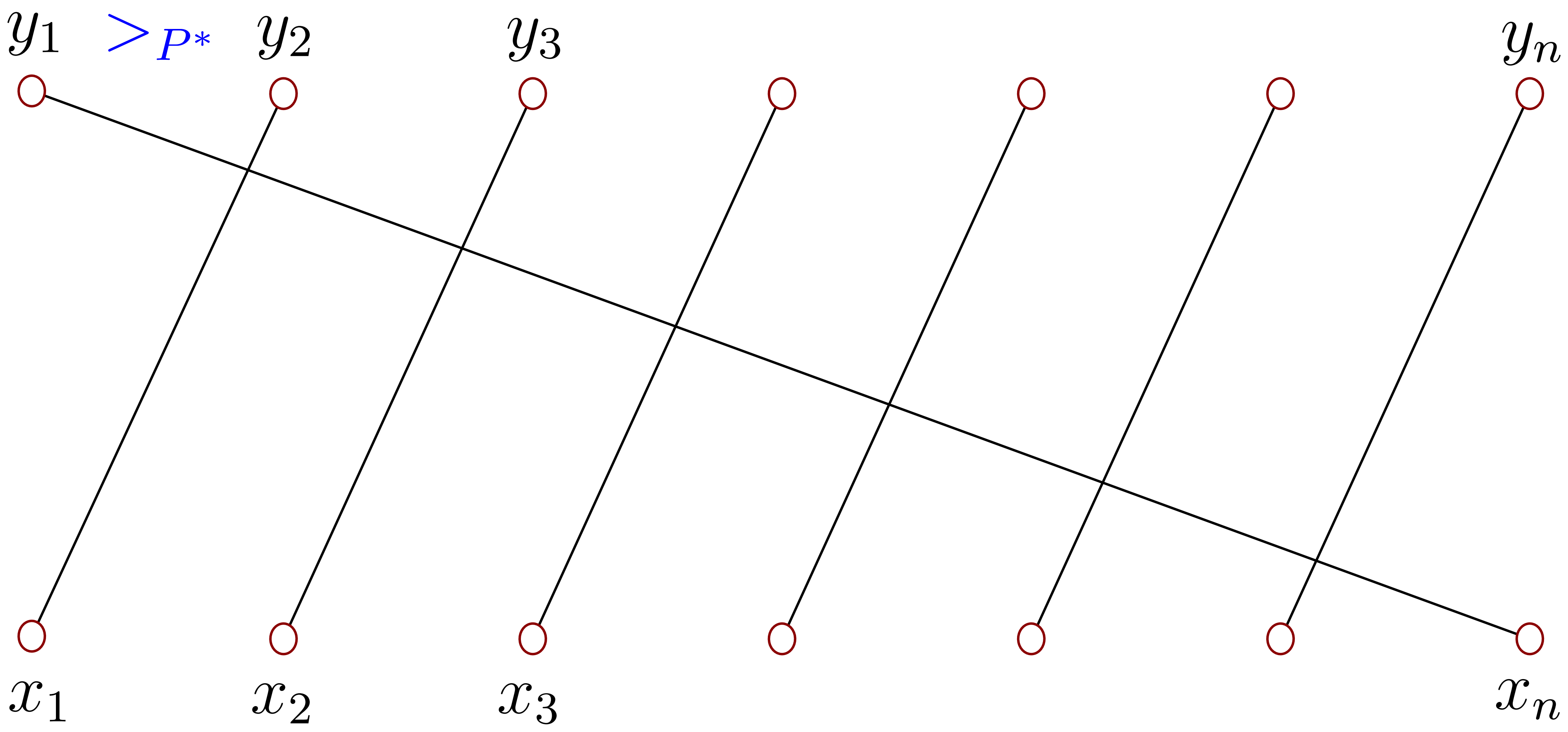


Claim. At least one of the two following statements hold:

(1) for all $y' \geq x$ in P, $y' \in T(u, B_i)$ (this implies $y' < y$ in P^*) R_1

(2) for all $x' \leq y$ in P, $x' \in T(v, B_i)$ (this implies $x < x'$ in P^*) R_2

Now we prove that R_1 is reversible. Assume it is not, then it has an alternating cycle:

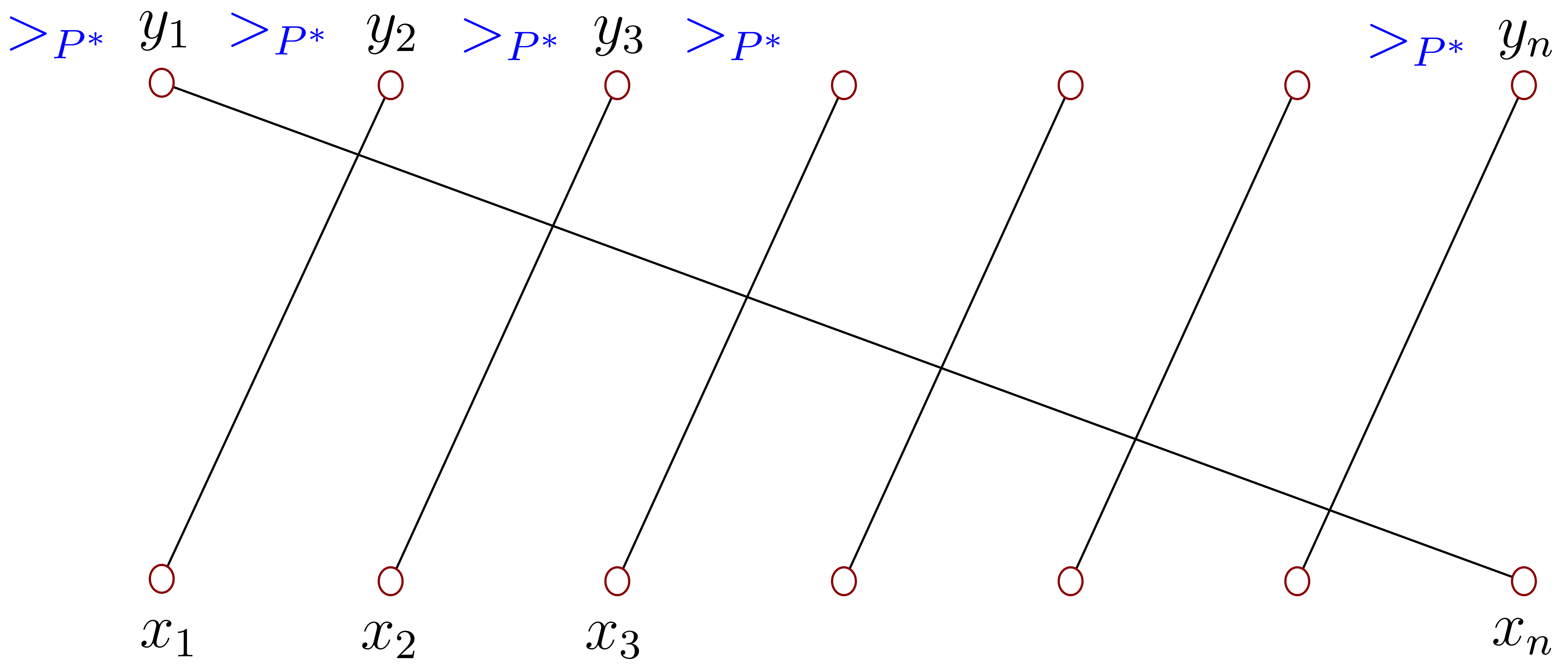


Claim. At least one of the two following statements hold:

(1) for all $y' \geq x$ in P, $y' \in T(u, B_i)$ (this implies $y' < y$ in P^*) R_1

(2) for all $x' \leq y$ in P, $x' \in T(v, B_i)$ (this implies $x < x'$ in P^*) R_2

Now we prove that R_1 is reversible. Assume it is not, then it has an alternating cycle:

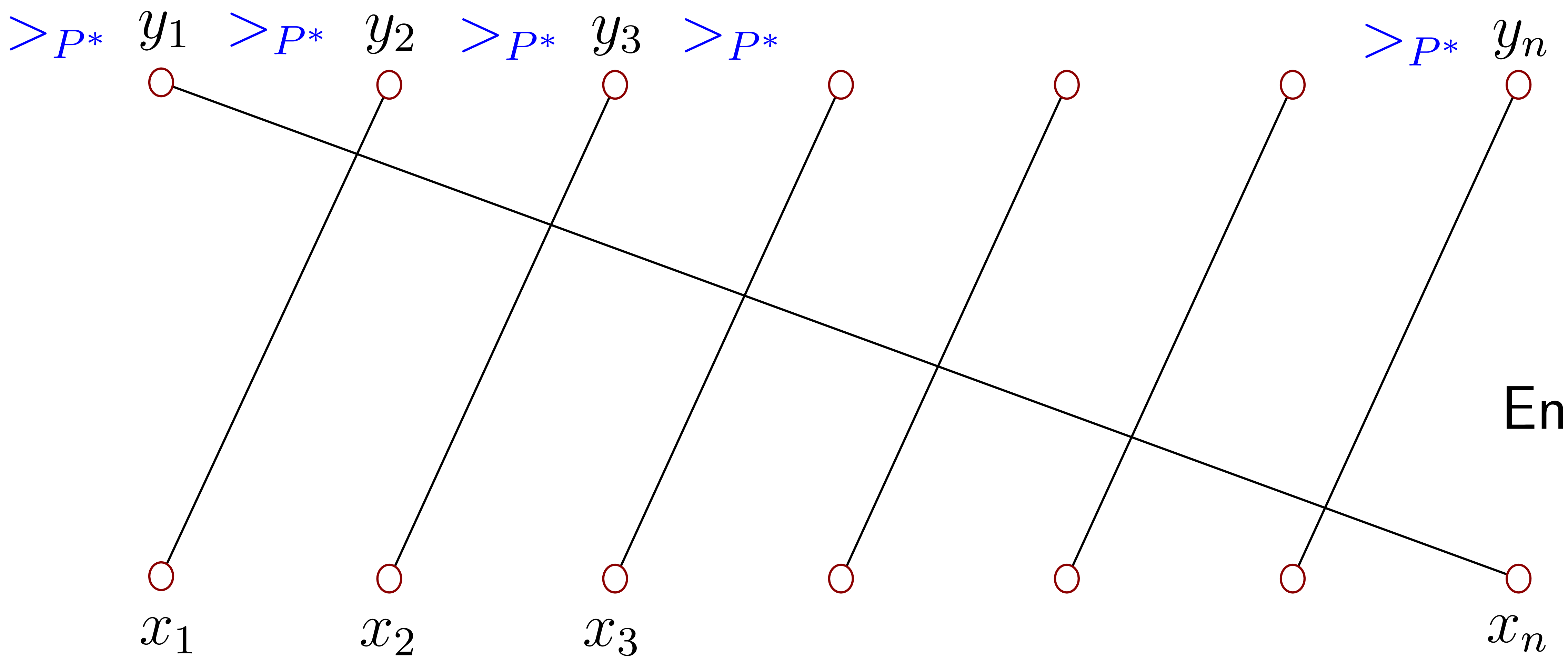


Claim. At least one of the two following statements hold:

(1) for all $y' \geq x$ in P, $y' \in T(u, B_i)$ (this implies $y' < y$ in P^*) $\quad R_1$

(2) for all $x' \leq y$ in P, $x' \in T(v, B_i)$ (this implies $x < x'$ in P^*) $\quad R_2$

Now we prove that R_1 is reversible. Assume it is not, then it has an alternating cycle:



End of proof $\square$

Using the *Product Ramsey Theorem*, authors prove that for any $d \geq 1$ there are posets P_d s.t. every block of P_d has dimension at most d, but $dim(P_d) = d + 2$

Using the *Product Ramsey Theorem*, authors prove that for any $d \geq 1$ there are posets P_d s.t. every block of P_d has dimension at most d, but $dim(P_d) = d + 2$

Product Ramsey Theorem. For every 4-tuple (r, d, k, m) of positive integers with $m \geq k$, there is an integer $n_0 \geq k$ s.t. if we have d set X_i and $|X_i| \geq n_0$ for every $i = 1, 2, ..., d$, then whenever we have a coloring ϕ which assigns to each k^d-grid g in $X_1 \times X_2 \times ... \times X_d$ a color $\phi(g)$ from a set R of r colors, then there is a color $\alpha \in R$, and there are m-element subsets $H_1, ..., H_d$ of $X_1, ..., X_d$ respectively, s. t. $\phi(g) = \alpha$ for every k^d grid in $H_1 \times ... \times H_d$

Thank you!