Critically paintable, choosable or colorable graphs

Grzegorz Gawryał

Jagiellonian University

Thursday 30 March, 2023

Based on A. Riasat and U. Schauz publication

Paintability

- A game on a graph between Lister and Painter. Before the game, both players know the graph and some function $l : V \to \{0, 1, \dots\}$.

Paintability

- A game on a graph between Lister and Painter. Before the game, both players know the graph and some function $l : V \to \{0, 1, \dots\}$.
- In i-th turn, Lister marks some uncolored vertices,

Paintability

- A game on a graph between Lister and Painter. Before the game, both players know the graph and some function $l : V \to \{0, 1, \dots\}$.
- In i-th turn, Lister marks some uncolored vertices,
- and then Painter colors some independent set of the marked vertices and i-th turn ends.

Paintability

- A game on a graph between Lister and Painter. Before the game, both players know the graph and some function $l : V \to \{0, 1, \dots\}$.
- In i-th turn, Lister marks some uncolored vertices,
- and then Painter colors some independent set of the marked vertices and i-th turn ends.
- Lister wins if some vertex has been marked more than $l(v)$ times.

Paintability

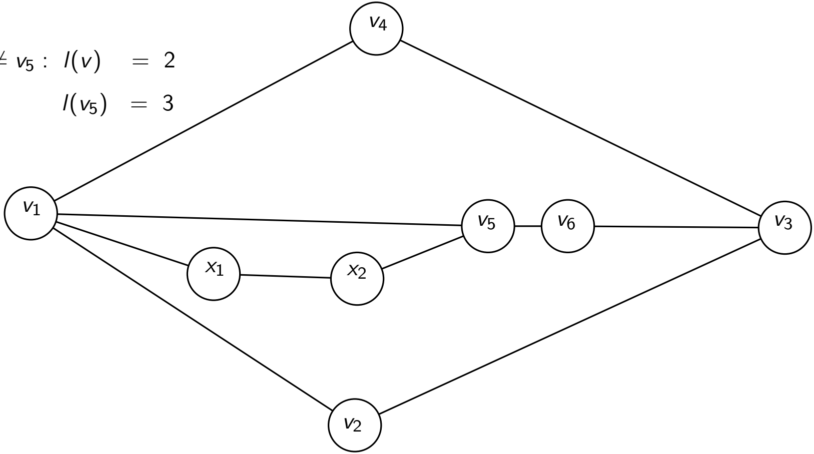
- A game on a graph between Lister and Painter. Before the game, both players know the graph and some function $l : V \to \{0, 1, \dots\}$.
- In i-th turn, Lister marks some uncolored vertices,
- and then Painter colors some independent set of the marked vertices and i-th turn ends.
- Lister wins if some vertex has been marked more than $l(v)$ times.
- Painter wins if all vertices have been colored before Lister wins.

Paintability

- A game on a graph between Lister and Painter. Before the game, both players know the graph and some function $l : V \to \{0, 1, \dots\}$.
- In i-th turn, Lister marks some uncolored vertices,
- and then Painter colors some independent set of the marked vertices and i-th turn ends.
- Lister wins if some vertex has been marked more than $l(v)$ times.
- Painter wins if all vertices have been colored before Lister wins.
- G is l-paintable if Painter has the winning strategy.

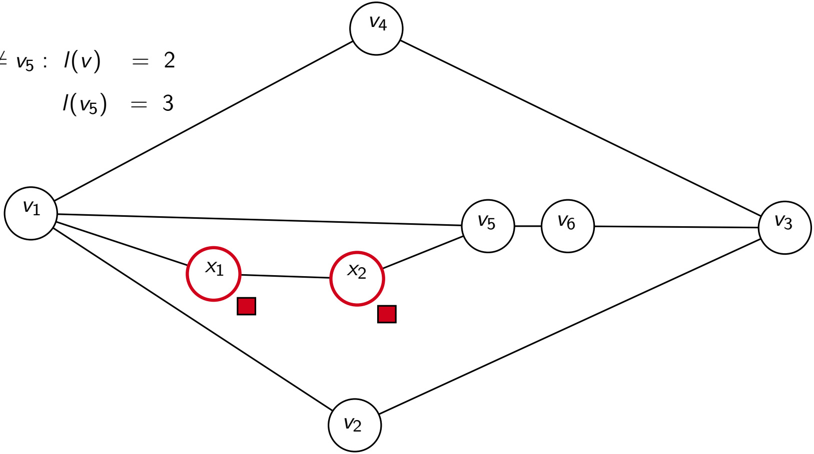
Paintability example

$$\begin{aligned} \forall v \neq v_5 : \ l(v) &= 2 \\ l(v_5) &= 3 \end{aligned}$$



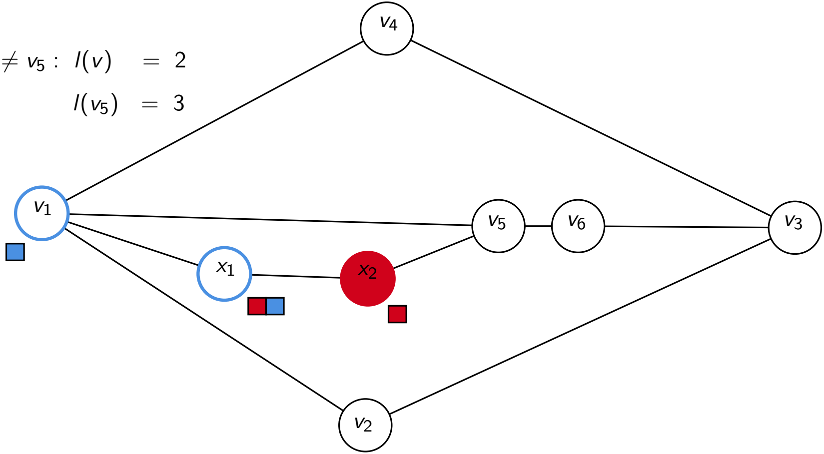
Paintability example

$$\begin{aligned} \forall v \neq v_5 : \ l(v) &= 2 \\ l(v_5) &= 3 \end{aligned}$$



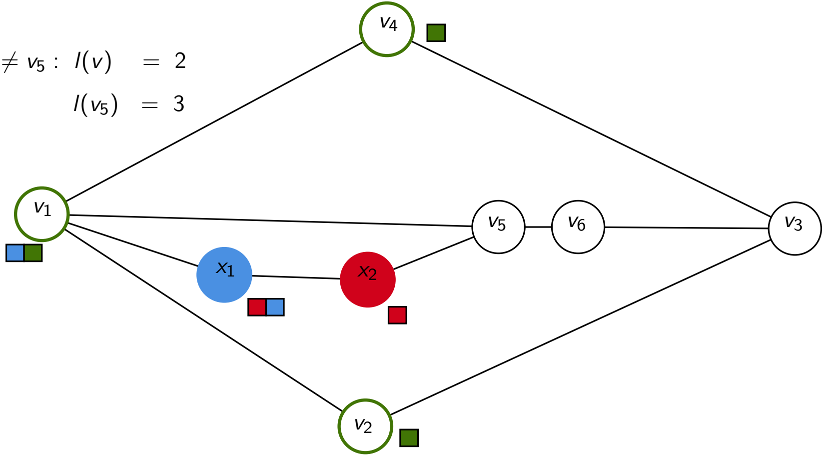
Paintability example

$$\forall v \neq v_5 : \; l(v) = 2$$
$$l(v_5) = 3$$



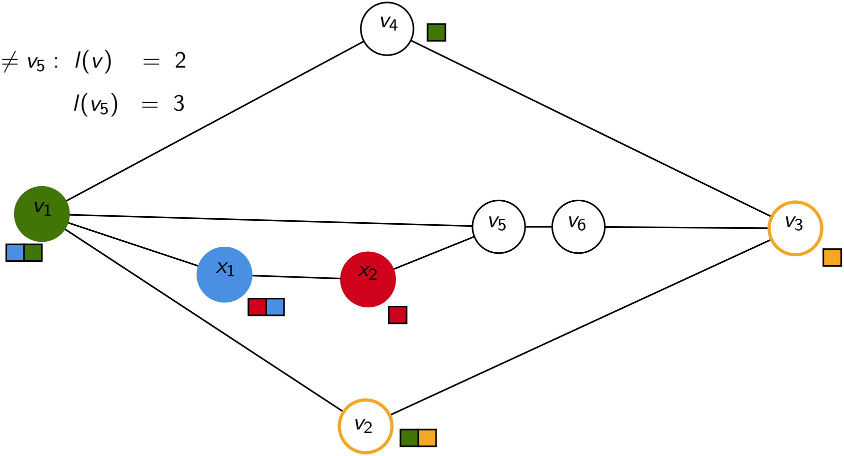
Paintability example

$$\forall v \neq v_5 : \; l(v) = 2$$
$$l(v_5) = 3$$



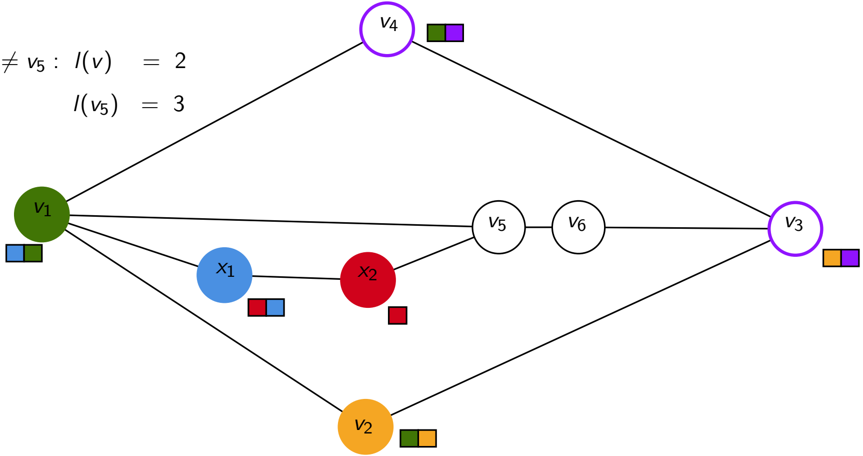
Paintability example

$$\forall v \neq v_5 : \; l(v) = 2$$
$$l(v_5) = 3$$



Paintability example

$$\forall v \neq v_5 : \; l(v) = 2$$
$$l(v_5) = 3$$



Relation between paintability

G is l-paintable $\implies$ G is l-choosable.

Relation between paintability

G is l-paintable $\implies$ G is l-choosable.

Moreover, if $\forall v : l(v) = k$, then:
G is l-choosable $\implies$ G is k-colorable.

Relation between paintability

G is l-paintable $\implies$ G is l-choosable.

Moreover, if $\forall v : l(v) = k$, then:
G is l-choosable $\implies$ G is k-colorable.

Implications in other directions don't hold generally.

Criticality

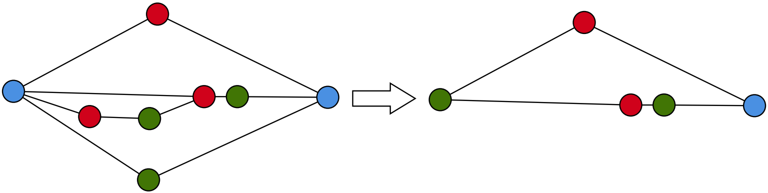
Graph G is almost l-paintable (choosable, colorable), if it is not l-paintable (choosable, colorable), but $\forall v : G \setminus \{v\}$ is.

Criticality

Graph G is almost l-paintable (choosable, colorable), if it is not l-paintable (choosable, colorable), but $\forall v : G \setminus \{v\}$ is.

Lemma

If G doesn't contain an almost l-paintable (colorable, choosable) induced subgraph, then it is l-paintable (colorable, choosable).



Strong version of Brooks' Theorem

Theorem (Hladký, Král, Schauz; 2010)

For any connected graph:

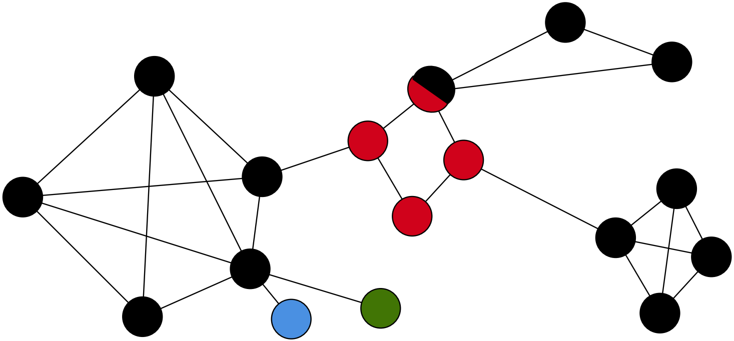
- *if it is a Gallai Tree, then it is not degree-choosable,*
- *otherwise it is degree-paintable.*

Strong version of Brooks' Theorem

Theorem (Hladký, Král, Schauz; 2010)

For any connected graph:

- *if it is a Gallai Tree, then it is not degree-choosable,*
- *otherwise it is degree-paintable.*



Cut Lemma

Lemma (Cut Lemma)

Let $G = (U \cup W, E)$*. Let* $\forall u \in U : \eta(u) = |N(u) \cap W|$*.*
If $G[W]$ *is* l*-paintable (choosable) and* $G[U]$ *is* $(l - \eta)$*-paintable (choosable), then* G *is* l*-paintable.*

Cut Lemma

Lemma (Cut Lemma)

Let $G = (U \cup W, E)$*. Let* $\forall u \in U : \eta(u) = |N(u) \cap W|$*.*
If $G[W]$ *is* l*-paintable (choosable) and* $G[U]$ *is* $(l - \eta)$*-paintable (choosable), then* G *is* l*-paintable.*

Consequence: Gallai Trees are almost degree-paintable and choosable.

Cut Lemma

Lemma (Cut Lemma)

Let $G = (U \cup W, E)$. Let $\forall u \in U : \eta(u) = |N(u) \cap W|$.
If $G[W]$ is l-paintable (choosable) and $G[U]$ is $(l - \eta)$-paintable (choosable), then G is l-paintable.

Consequence: Gallai Trees are almost degree-paintable and choosable.

Moreover, almost l-paintable (choosable) graphs satisify $\forall v : d(v) \geq l(v)$.

Low-degree subgraphs

For almost paintable (choosable) G, a low-degree subgraph is an induced subgraph on vertices for which $d(v) = l(v)$.

Low-degree subgraphs

For almost paintable (choosable) G, a low-degree subgraph is an induced subgraph on vertices for which $d(v) = l(v)$.

Lemma

The low-degree subgraph H of any almost paintable (choosable) graph G is a Gallai Forest.

Low-degree subgraphs

For almost paintable (choosable) G, a low-degree subgraph is an induced subgraph on vertices for which $d(v) = l(v)$.

Lemma

The low-degree subgraph H of any almost paintable (choosable) graph G is a Gallai Forest.

Proof: if some biconnected component B of H is neither clique nor cycle, then it is degree-paintable (choosable). Using almost-paintability, $G \setminus B$ is l-paintable, so from the cut lemma, G would also be.

Edge density lower bound

Lemma (Gallai, Kritische)

For a Gallai Tree $G = (V, E)$ different from $K_{\Delta+1}$ and with $\Delta \geq 3$:

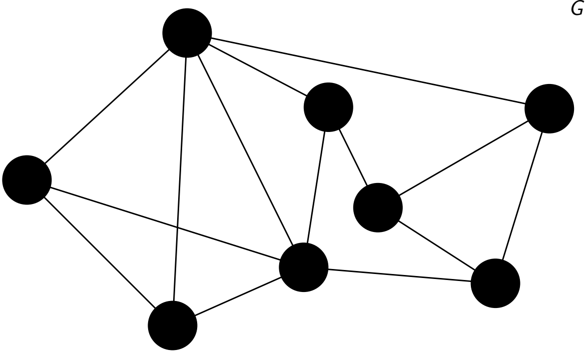
$$\frac{|E|}{|V|} < \frac{\Delta - 1}{2} + \frac{1}{\Delta}$$

Edge density lower bound

Lemma (Gallai, Kritische)

For a Gallai Tree $G = (V, E)$ different from $K_{\Delta+1}$ and with $\Delta \geq 3$:

$$\frac{|E|}{|V|} < \frac{\Delta - 1}{2} + \frac{1}{\Delta}$$



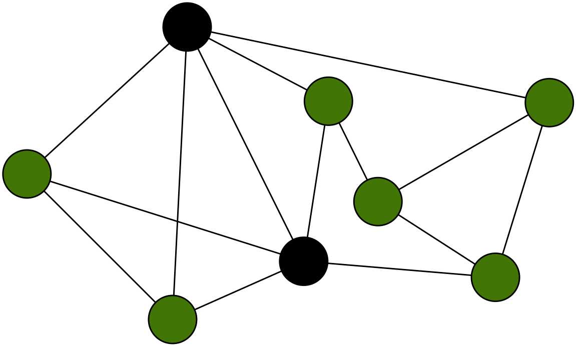
G – connected, non-complete

Edge density lower bound

Lemma (Gallai, Kritische)

For a Gallai Tree $G = (V, E)$ different from $K_{\Delta+1}$ and with $\Delta \geq 3$:

$$\frac{|E|}{|V|} < \frac{\Delta - 1}{2} + \frac{1}{\Delta}$$



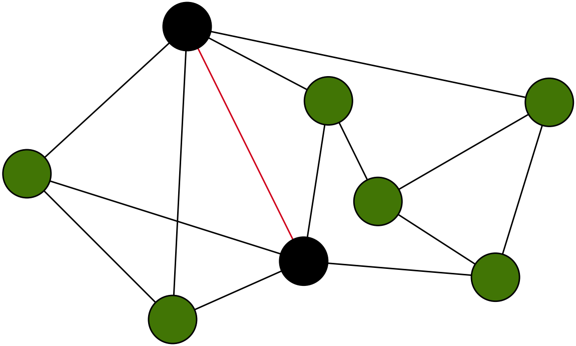
G – connected, non-complete
$H = G[\{u : d_G(u) = \delta\}]$

Edge density lower bound

Lemma (Gallai, Kritische)

For a Gallai Tree $G = (V, E)$ different from $K_{\Delta+1}$ and with $\Delta \geq 3$:

$$\frac{|E|}{|V|} < \frac{\Delta - 1}{2} + \frac{1}{\Delta}$$



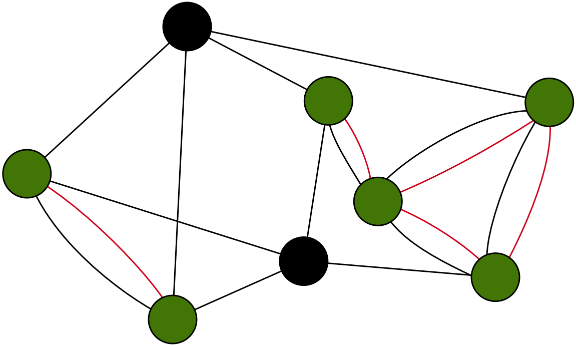
G − connected, non-complete
$H = G[\{u : d_G(u) = \delta\}]$
$|E(G)| \geq |E(G)| - |E(G \setminus H)|$

Edge density lower bound

Lemma (Gallai, Kritische)

For a Gallai Tree $G = (V, E)$ different from $K_{\Delta+1}$ and with $\Delta \geq 3$:

$$\frac{|E|}{|V|} < \frac{\Delta - 1}{2} + \frac{1}{\Delta}$$



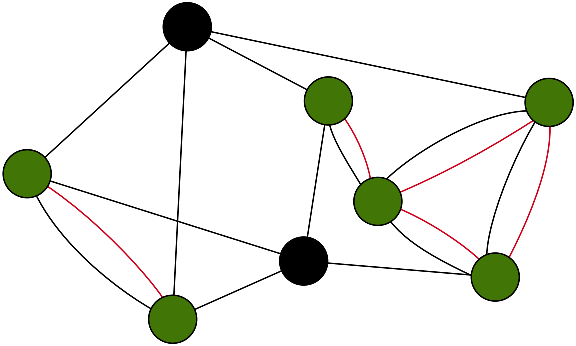
G – connected, non-complete
$H = G[\{u : d_G(u) = \delta\}]$
$|E(G)| \geq |E(G)| - |E(G \setminus H)|$
$= \delta|V(H)| - |E(H)|$

Edge density lower bound

Lemma (Gallai, Kritische)

For a Gallai Tree $G = (V, E)$ different from $K_{\Delta+1}$ and with $\Delta \geq 3$:

$$\frac{|E|}{|V|} < \frac{\Delta - 1}{2} + \frac{1}{\Delta}$$



G – connected, non-complete

$H = G[\{u : d_G(u) = \delta\}]$

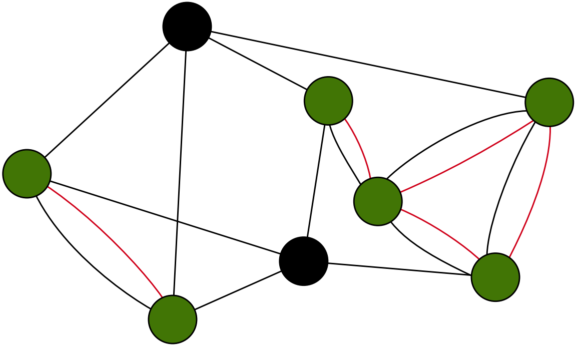
$$\begin{aligned} |E(G)| &\geq |E(G)| - |E(G \setminus H)| \\ &= \delta |V(H)| - |E(H)| \\ &> |V(H)| \left(\delta - \frac{\Delta_H - 1}{2} + \frac{1}{\Delta_H}\right) \end{aligned}$$

Edge density lower bound

Lemma (Gallai, Kritische)

For a Gallai Tree $G = (V, E)$ different from $K_{\Delta+1}$ and with $\Delta \geq 3$:

$$\frac{|E|}{|V|} < \frac{\Delta - 1}{2} + \frac{1}{\Delta}$$



G – connected, non-complete

$H = G[\{u : d_G(u) = \delta\}]$

$$\begin{aligned} |E(G)| &\geq |E(G)| - |E(G \setminus H)| \\ &= \delta|V(H)| - |E(H)| \\ &> |V(H)| \left(\delta - \frac{\Delta_H - 1}{2} + \frac{1}{\Delta_H}\right) \\ &\geq |V(H)| \left(\delta - \frac{\delta - 1}{2} + \frac{1}{\delta}\right) \end{aligned}$$

Edge density - cont.

On the other side:

$$2|E| \geq (\delta + 1)|V(G \setminus H)| + \delta|V(H)| = (\delta + 1)|V| - |V(H)|$$

Edge density - cont.

On the other side:

$$2|E| \geq (\delta + 1)|V(G \setminus H)| + \delta|V(H)| = (\delta + 1)|V| - |V(H)|$$

Combined with a previous result:

$$2\frac{|E|}{|V|} > \delta + \frac{\delta - 2}{\delta^2 + 2\delta - 2}$$

Edge density - cont.

On the other side:

$$2|E| \geq (\delta+1)|V(G \setminus H)| + \delta|V(H)| = (\delta+1)|V| - |V(H)|$$

Combined with a previous result:

$$2\frac{|E|}{|V|} > \delta + \frac{\delta-2}{\delta^2+2\delta-2}$$

Lemma

If $G \neq K_{k+1}$ is almost l-paintable (l-choosable, l-colorable), where $k := min(l(v)) \geq 3$, then:

$$2\frac{|E|}{|V|} > k + \frac{k-2}{k^2+2k-2}$$

Graphs on surfaces

Graphs on surfaces

Lemma (Euler's formula)

For any connected graph G drawn on a surface with a genus g:
$2 - g = |V(G)| - |E(G)| + |F(G)|$
Moreover:
$2 - g \leq |V(G)| - \frac{1}{3}|E(G)|$

Bounding degeneracy

$$6(2 - g) \leq 6|V| - 2|E| \leq (6 - \delta)|V|$$

Bounding degeneracy

$6(2-g) \leq 6|V| - 2|E| \leq (6-\delta)|V|$

So, when $\delta \geq 6$:

$0 \geq 6(2-g) + (\delta - 6)|V| \geq 6(2-g) + (\delta - 6)(\delta + 1)$

Bounding degeneracy

$6(2-g) \leq 6|V| - 2|E| \leq (6-\delta)|V|$

So, when $\delta \geq 6$:

$0 \geq 6(2-g) + (\delta-6)|V| \geq 6(2-g) + (\delta-6)(\delta+1)$

$$\delta \leq \frac{5+\sqrt{1+24g}}{2} < \left\lfloor \frac{7+\sqrt{1+24g}}{2} \right\rfloor := H(g), \text{ the Heawood number}$$

Bounding degeneracy

$$6(2-g) \leq 6|V| - 2|E| \leq (6-\delta)|V|$$

So, when $\delta \geq 6$:

$$0 \geq 6(2-g) + (\delta-6)|V| \geq 6(2-g) + (\delta-6)(\delta+1)$$

$$\delta \leq \frac{5+\sqrt{1+24g}}{2} < \left\lfloor \frac{7+\sqrt{1+24g}}{2} \right\rfloor := H(g), \text{ the Heawood number}$$

And for $\delta < 6$, but $g \geq 1$: $\delta \leq 5 < H(1) \leq \cdots \leq H(g)$

Bounding degeneracy

$$6(2-g) \leq 6|V| - 2|E| \leq (6-\delta)|V|$$

So, when $\delta \geq 6$:

$$0 \geq 6(2-g) + (\delta - 6)|V| \geq 6(2-g) + (\delta-6)(\delta+1)$$

$$\delta \leq \frac{5+\sqrt{1+24g}}{2} < \left\lfloor \frac{7+\sqrt{1+24g}}{2} \right\rfloor := H(g), \text{ the Heawood number}$$

And for $\delta < 6$, but $g \geq 1$: $\delta \leq 5 < H(1) \leq \cdots \leq H(g)$

So, the graph is $H(g) - 1$-degenerate, and from the cut lemma also $H(g)$-paintable.

Heawood's Map-Coloring Theorem

Theorem

A graph G on a surface with genus $g \geq 1$ is $H(g)$-paintable, choosable, and colorable. Except for the Klein Bottle, this is the best possible bound.

Heawood's Map-Coloring Theorem

Theorem

A graph G on a surface with genus $g \geq 1$ is $H(g)$-paintable, choosable, and colorable. Except for the Klein Bottle, this is the best possible bound.

The lower bound holds, because $K_{H(g)}$ can be embedded on all surfaces of genus g, except for the Klein Bottle.

Heawood's Map-Coloring Theorem

Theorem

A graph G on a surface with genus $g \geq 1$ is $H(g)$-paintable, choosable, and colorable. Except for the Klein Bottle, this is the best possible bound.

The lower bound holds, because $K_{H(g)}$ can be embedded on all surfaces of genus g, except for the Klein Bottle.

If the graph doesn't contain $K_{H(g)}$, then the number of required colors can be decreased by 1 for list coloring.

Heawood's Map-Coloring Theorem

Theorem

A graph G on a surface with genus $g \geq 1$ is $H(g)$-paintable, choosable, and colorable. Except for the Klein Bottle, this is the best possible bound.

The lower bound holds, because $K_{H(g)}$ can be embedded on all surfaces of genus g, except for the Klein Bottle.

If the graph doesn't contain $K_{H(g)}$, then the number of required colors can be decreased by 1 for list coloring.

Likely, it's also true for paintability – proven for all g except $g \in \{1, 3\}$.

Critical graphs on surfaces

Let G be an almost-paintable graph on a surface with genus g.

Recall, that $6(2-g) \leq 6|V| - 2|E| \leq (6-\delta)|V|$.

Critical graphs on surfaces

Let G be an almost-paintable graph on a surface with genus g.

Recall, that $6(2-g) \leq 6|V| - 2|E| \leq (6-\delta)|V|$.

If $\delta \geq 7$, then, using the fact, that $\forall v \in V : l(v) \leq d(v)$:

$$|V| \leq \frac{6(g-2)}{\delta - 6} \leq \frac{6(g-2)}{\min(l) - 6}$$

Critical graphs on surfaces

Let G be an almost-paintable graph on a surface with genus g.

Recall, that $6(2-g) \leq 6|V| - 2|E| \leq (6-\delta)|V|$.

If $\delta \geq 7$, then, using the fact, that $\forall v \in V : l(v) \leq d(v)$:

$$|V| \leq \frac{6(g-2)}{\delta - 6} \leq \frac{6(g-2)}{\min(l) - 6}$$

And if $\min(l) = 6$ and $G \neq K_7$:

$$6(2-g) \leq 6|V| - 2|E| \leq \left(6 - \left(6 + \frac{6-2}{6^2 + 2 \cdot 6 - 2}\right)\right)|V| \leq \frac{2}{23}|V|$$

Critical graphs on surfaces

Let G be an almost-paintable graph on a surface with genus g.

Recall, that $6(2-g) \leq 6|V| - 2|E| \leq (6-\delta)|V|$.

If $\delta \geq 7$, then, using the fact, that $\forall v \in V : l(v) \leq d(v)$:

$$|V| \leq \frac{6(g-2)}{\delta - 6} \leq \frac{6(g-2)}{\min(l) - 6}$$

And if $\min(l) = 6$ and $G \neq K_7$:

$$6(2-g) \leq 6|V| - 2|E| \leq \left(6 - \left(6 + \frac{6-2}{6^2 + 2 \cdot 6 - 2}\right)\right)|V| \leq \frac{2}{23}|V|$$

In both cases $|V(G)| < 69(g-2)$, so if $\min(l) \geq 6$, then there are only finitely many pairs (G, l) of almost l-paintable (choosable) graphs.

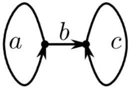
Almost 2-paintable and choosable graphs

If G has a vertex v with $d(v) = 1, l(v) \geq 2$, then G is l-paintable iff $G \setminus \{v\}$ is so.

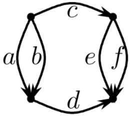
Almost 2-paintable and choosable graphs

If G has a vertex v with $d(v) = 1, l(v) \geq 2$, then G is l-paintable iff $G \setminus \{v\}$ is so.

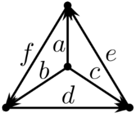
Assume we have no 1-vertices. Let C_n be a cycle of length n and:



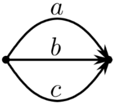
$\text{O-O}_{a,b,c}$



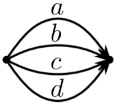
$\text{O=O}_{a,b,c,d,e,f}$



$K_4^{a,b,c,d,e,f}$



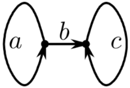
$\Theta_{a,b,c}$



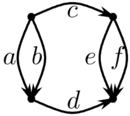
$\Theta_{a,b,c,d}$

If G has a vertex v with $d(v) = 1, l(v) \geq 2$, then G is l-paintable iff $G \setminus \{v\}$ is so.

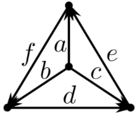
Assume we have no 1-vertices. Let C_n be a cycle of length n and:



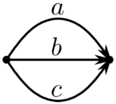
$\text{O-O}_{a,b,c}$



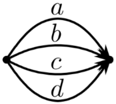
$\text{O=O}_{a,b,c,d,e,f}$



$K_4^{a,b,c,d,e,f}$



$\Theta_{a,b,c}$



$\Theta_{a,b,c,d}$

We'll show, that:
$\mathcal{T}_{ch} := C_{2a} \cup \Theta_{2,2,2\alpha}$ and $\mathcal{T}_p := C_{2\alpha} \cup \{\Theta_{2,2,2}\}$
are exactly the classes of 2-choosable (paintable) graphs.

Proof sketch:

1. Taking vertex minors (taking subgraph, then contracting vertices = contracting all edges incident to it) preserves 2-choosability and 2-paintability.

2-choosability and paintability identification

Proof sketch:

1. Taking vertex minors (taking subgraph, then contracting vertices = contracting all edges incident to it) preserves 2-choosability and 2-paintability.
2. G is either $C_\alpha \vee \Theta_{\alpha,\beta,\gamma}$ or contains $C_3, K_{3,3}$, , , or a subdivided K_4.

2-choosability and paintability identification

Proof sketch:

1. Taking vertex minors (taking subgraph, then contracting vertices = contracting all edges incident to it) preserves 2-choosability and 2-paintability.
2. G is either $C_\alpha \vee \Theta_{\alpha,\beta,\gamma}$ or contains $C_3, K_{3,3}$, , , or a subdivided K_4.
3. Identify minimal elements of $G \setminus \mathcal{T}_{ch}$ w.r.t. induced subgraphs.

2-choosability and paintability identification

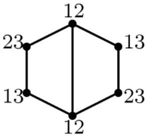
Proof sketch:

1. Taking vertex minors (taking subgraph, then contracting vertices = contracting all edges incident to it) preserves 2-choosability and 2-paintability.
2. G is either $C_\alpha \vee \Theta_{\alpha,\beta,\gamma}$ or contains $C_3, K_{3,3}$, , , or a subdivided K_4.
3. Identify minimal elements of $G \setminus \mathcal{T}_{ch}$ w.r.t. induced subgraphs.
4. Identify minimal elements w.r.t usual subgraphs.

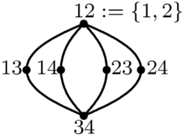
2-choosability and paintability identification

Proof sketch:

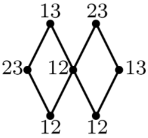
1. Taking vertex minors (taking subgraph, then contracting vertices = contracting all edges incident to it) preserves 2-choosability and 2-paintability.
2. G is either $C_\alpha \vee \Theta_{\alpha,\beta,\gamma}$ or contains $C_3, K_{3,3}$, O-O, O-O, or a subdivided K_4.
3. Identify minimal elements of $G \setminus \mathcal{T}_{ch}$ w.r.t. induced subgraphs.
4. Identify minimal elements w.r.t usual subgraphs.
5. Identify minimal elements w.r.t vertex minors.



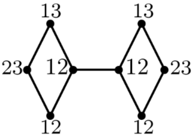
$\Theta_{1,3,3}$



$K_{2,4} = \Theta_{2,2,2,2}$

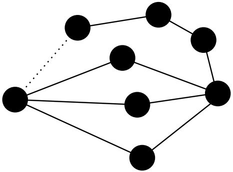


O-O$_{4,0,4}$

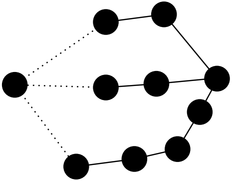


O-O$_{4,1,4}$

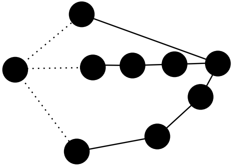
Almost 2-choosable graphs w.r.t. vertex deletion:



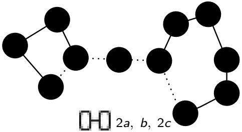
$\Theta_{2a,\ 2,\ 2,\ 2}$



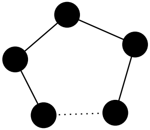
$\Theta_{2a+1,\ 2b+1,\ 2c+1}$



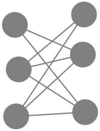
$\Theta_{2a,\ 2b+4,\ 2c+4}$



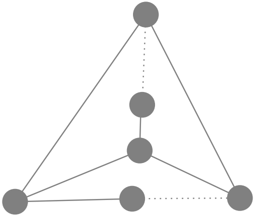
$\mathrm{O{-}O}_{2a,\ b,\ 2c}$



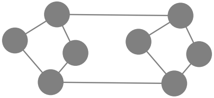
C_{2a+1}



$K_{3,3}$

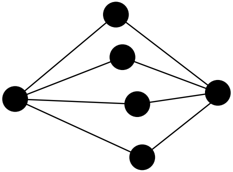


$K_4^{2a,1,1,2d,1,1}$

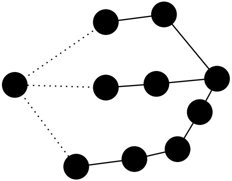


$\mathrm{O{=}O}_{2,2,1,1,2,2}$

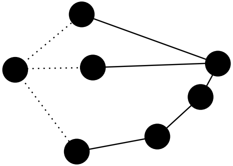
Almost 2-paintable graphs w.r.t. vertex deletion:



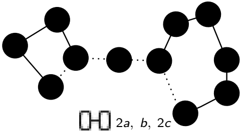
$K_{2,4}$



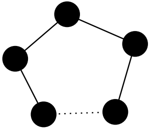
$\Theta_{2a+1,\ 2b+1,\ 2c+1}$



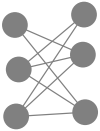
$\Theta_{2a,\ 2b,\ 2c+4}$



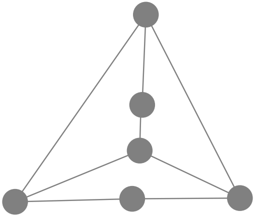
⊶ $_{2a,\ b,\ 2c}$



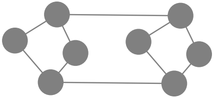
C_{2a+1}



$K_{3,3}$



$K_4^{2,1,1,2,1,1}$



⊶ ^{2,2,1,1,2,2}