Decomposing 4-connected planar triangulations into two trees and one path

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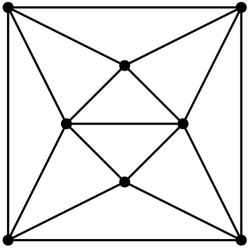
April 20, 2023

INTRODUCTION

Graph decomposition



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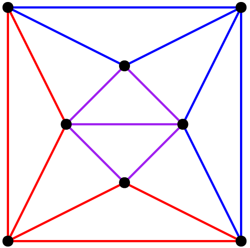

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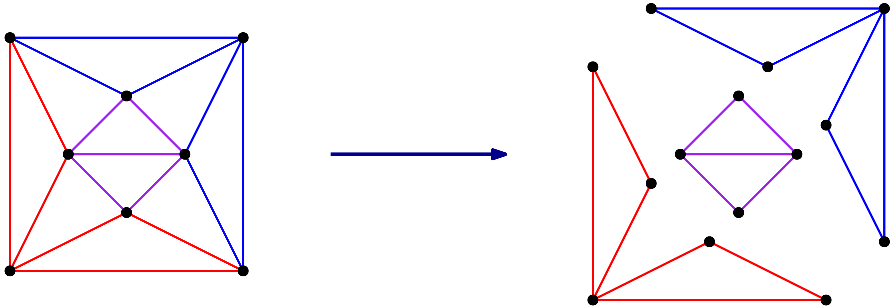

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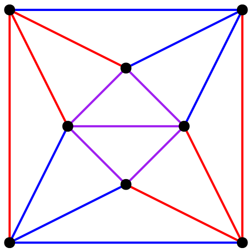

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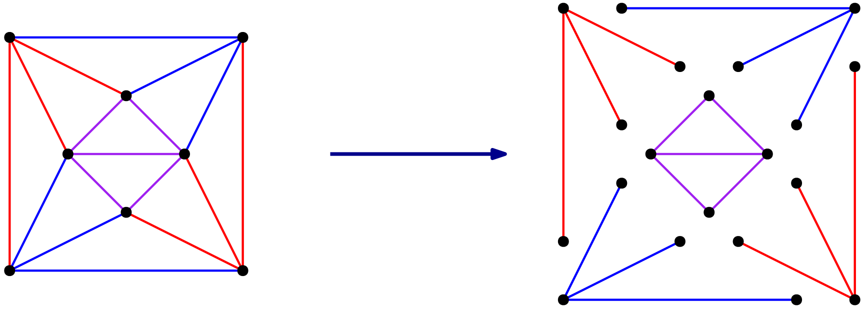

Arboricity

Arboricity of graph G is defined as follows:

$$a(G) = \max_{S \subseteq V(G); |S| \geq 2} \frac{|E(S)|}{|S| - 1}$$

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Nash-Williams Theorem

For every graph G, the minimal number of forests that G can be divided into is equal to $\lceil a(G) \rceil$.

Observation

Any planar graph G with n vertices has at most $3n - 6$ edges.
Therefore $a(G) \leq 3$, so every planar graph can be divided into 3 forests.

A graph G is (k, d)-*decomposable* if it can be decomposed into k forests and one **subgraph** of maximum degree d.

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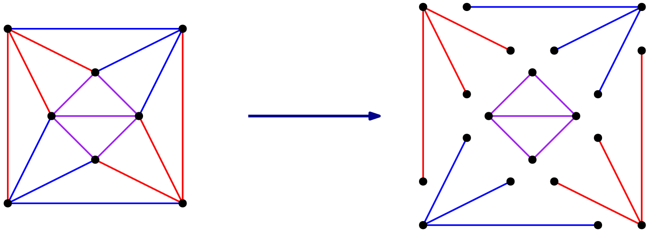

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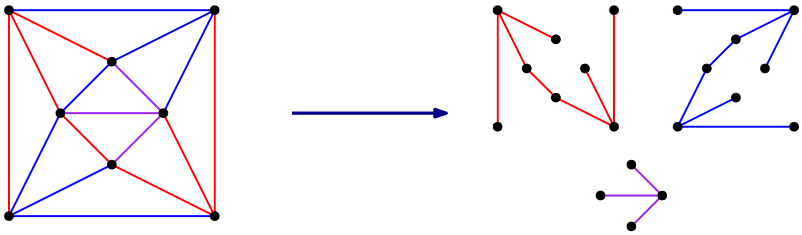

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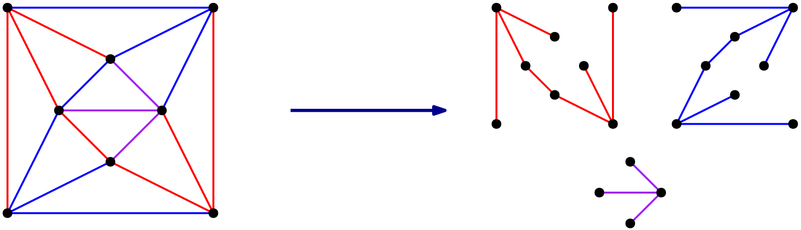

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Nine Dragon Tree Theorem:
If for any graph G, there exist positive integers k and d such that:

$$a(G) \leq k + \frac{d}{k+d+1},$$

then G is $(k, d)^{\star}$-*decomposable*.

Notice that $\lceil a(G) \rceil \leq k + 1$.

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- Planar graphs with girth at least 8 are $(1,1)^{\star}$-*decomposable*
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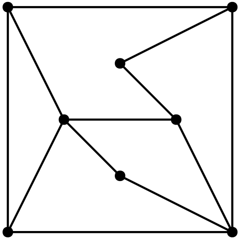
For every planar graph G with girth at least g we have $a(G) \leq \frac{g}{g-2}$

Planar triangulation



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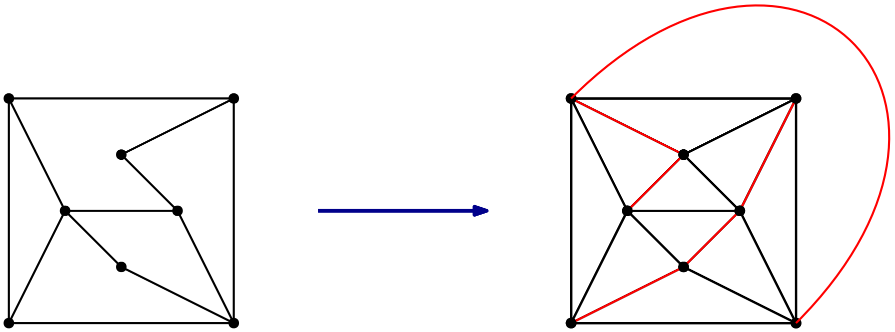

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Every planar graph is $(2, 4)^{\star}$-*decomposable*.
Some planar graphs are not $(2, 3)$-*decomposable*.

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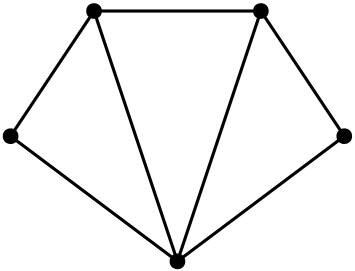

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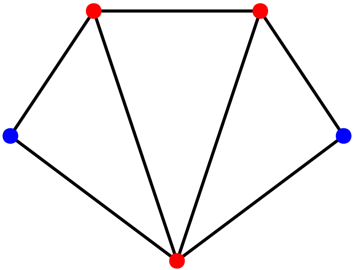

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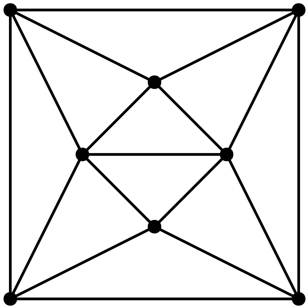
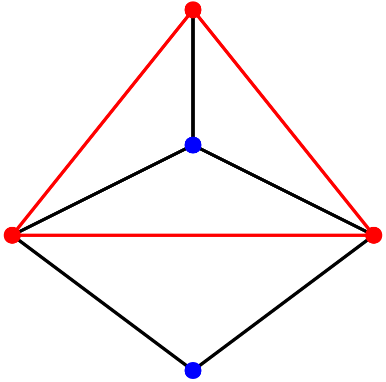


Separating triangle



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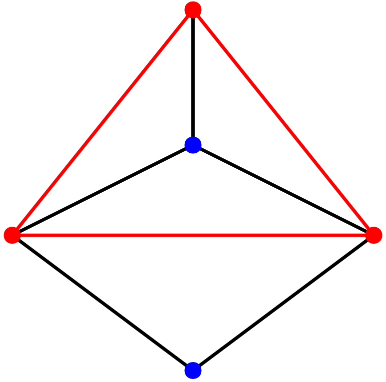

Planar triangulation is **4-connected** if and only if it does not have any **separating triangles**.

RESULTS

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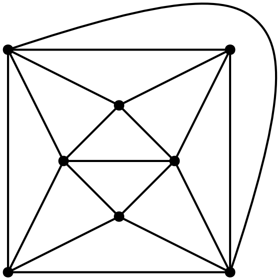

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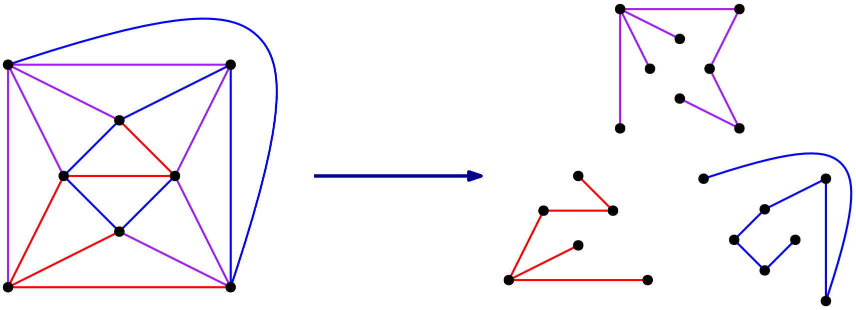

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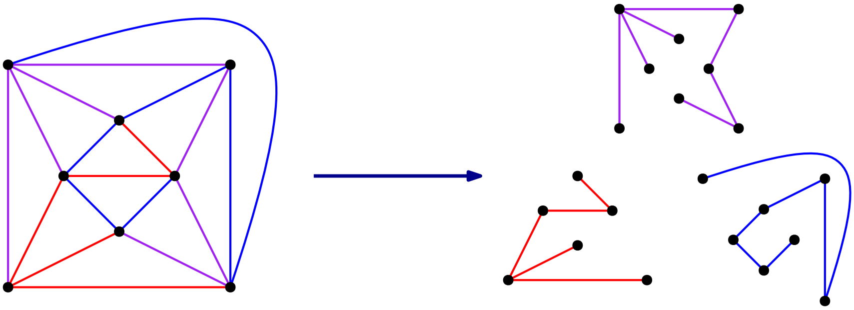

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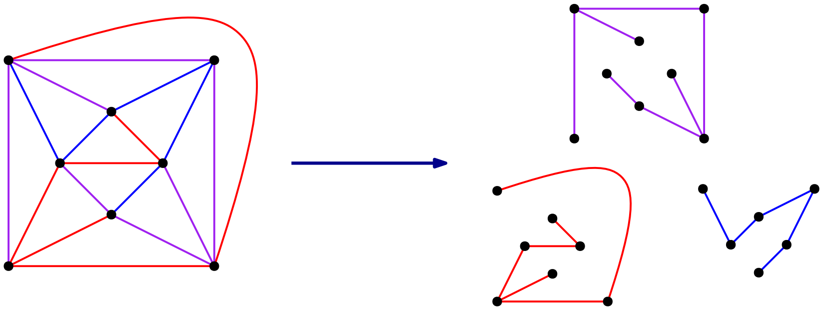

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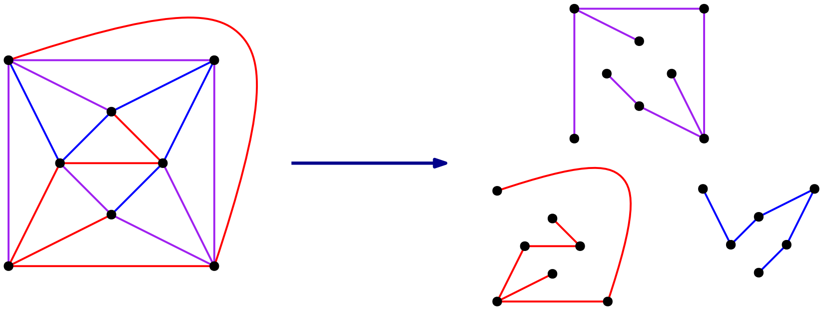

Moreover some Hamiltonian planar triangulations are not $(2, 2)$-*decomposable*.

Theorem 3

Every **4-connected** planar triangulation G decomposes into two trees and one Hamiltionian path.

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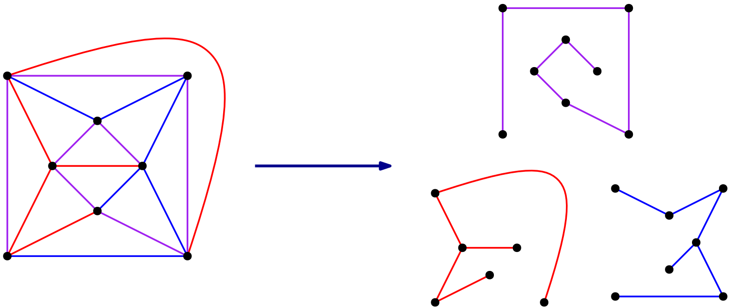

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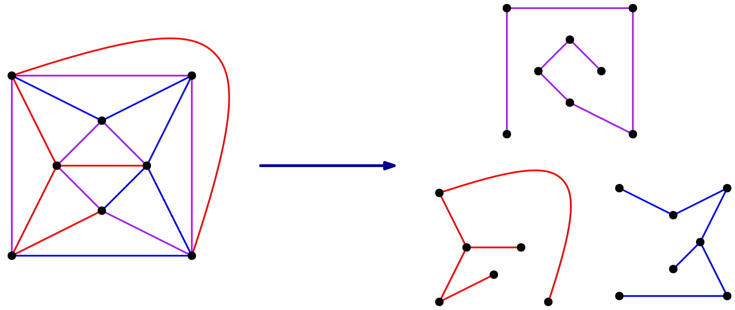

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Every n-vertex $(2, 1)$-*decomposable* graph decomposes into 2 forests and one matching, therefore it has at most $2(n-1) + n/2$ edges.

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For $n \geq 9$:

$$3n\text{-}6 > 2(n\text{-}1) + n/2,$$

leading to contradiction

OPEN QUESTIONS

THANKS FOR THE ATTENTION