Improved lower bounds on the number of edges in list critical and online list critical graphs

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Introduction

A *k-coloring* of a graph G is a function $\pi : V(G) \to \{1, \ldots, k\}$ such that $\pi(x) \neq \pi(y)$ for each edge xy.
$\chi(G)$ is the least integer k such that G is k-colorable.

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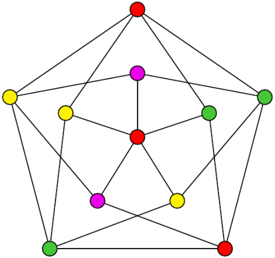
A graph G is *k-critical* if G is not $(k-1)$-colorable, but every proper subgraph of G is.



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A graph G is *L-colorable* if it has a proper coloring using colors from a set of lists L.
For $f : V(G) \to \mathbb{N}$, a list assignment L is an f-assignment if $|L(v)| = f(v)$ for each $v \in V(G)$.

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G is *k-choosable* if G is f-choosable for $|f(v)| = k$.

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G is *k-choosable* if G is f-choosable for $|f(v)| = k$.

A graph G is *k-list-critical* if there exists L with $|L(v)| = k - 1$ such that G is not L-colorable, but every proper subgraph of G is L-colorable.

Main result

If G is k-critical then $\delta(G) \geq k-1$, so $2\,\|G\| \geq (k-1)|G|$.

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$$\|G\| \geq \left\lceil \frac{(k+1)(k-2)|G| - k(k-3)}{2(k-1)} \right\rceil .$$

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We prove that every k-list-critical graph ($k \geq 7$) on $n \geq k+2$ vertices has at least

$$\frac{1}{2}\left(k - 1 + \frac{k-3}{(k-c)(k-1) + k - 3}\right) n$$

edges where

$$c = (k-3)\left(\frac{1}{2} - \frac{1}{(k-1)(k-2)}\right).$$

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If $\delta(G) \geq k$ the bound holds, so we may assume $\delta(G) = k-1$.

History of results

	k-Critical G				k-ListCritical G	
k	Gallai $d(G) \geq$	Kriv $d(G) \geq$	KS $d(G) \geq$	KY $d(G) \geq$	KS $d(G) \geq$	Here $d(G) \geq$
4	3.0769	3.1429	—	3.3333	—	—
5	4.0909	4.1429	—	4.5000	—	**4.0984**
6	5.0909	5.1304	5.0976	5.6000	—	**5.1053**
7	6.0870	6.1176	6.0990	6.6667	—	**6.1149**
8	7.0820	7.1064	7.0980	7.7143	—	**7.1128**
9	8.0769	8.0968	8.0959	8.7500	8.0838	**8.1094**
10	9.0722	9.0886	9.0932	9.7778	9.0793	**9.1055**
15	14.0541	14.0618	14.0785	14.8571	14.0610	**14.0864**
20	19.0428	19.0474	19.0666	19.8947	19.0490	**19.0719**

Table: History of lower bounds on the average degree $d(G)$ of k-critical and k-list-critical graphs G.

Alon-Tarsi

- A subgraph H of a directed multigraph is called *Eulerian* if $d_H^-(v) = d_H^+(v)$ for every $v \in V(H)$.

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- H is *even* if $\|H\|$ is even and *odd* otherwise.
- Let $EE(D)$ be the number of even spanning Eulerian subgraphs of D.
- Let $EO(D)$ be the number of odd spanning Eulerian subgraphs of D.
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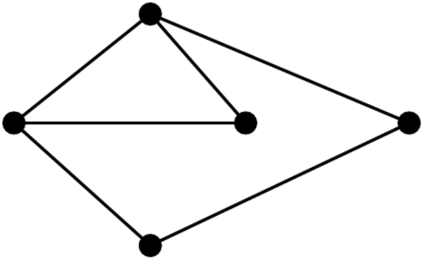
A graph G is *f-Alon-Tarsi* or f-AT if G has an orientation D where $f(v) \geq d_D^+(v) + 1$ for all $v \in V(D)$ and $EE(D) \neq EO(D)$.

Alon-Tarsi



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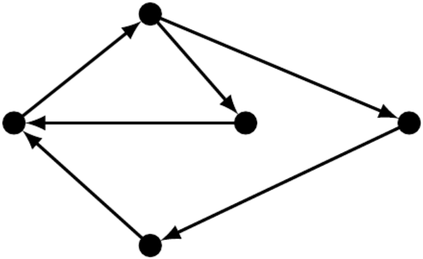

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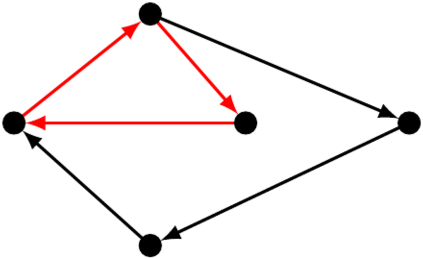

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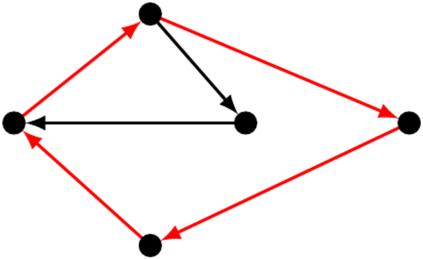

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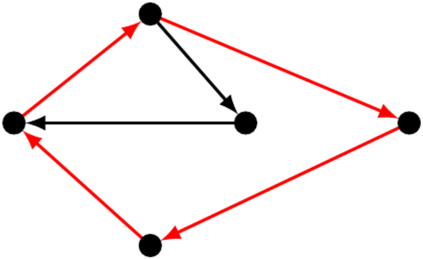

Lemma

If a graph G is f-AT, then G is f-choosable.

Alon-Tarsi

Combinatorial Nullstellensatz

Let $f(x_1, \ldots, x_n)$ be a polynomial over $\mathbb{Z}$. Suppose that the coefficient of the monomial $x_1^{k_1} \cdots x_n^{k_n}$ in f is nonzero and $k_1 + \ldots + k_n$ is equal to the total degree of f. If $A_1, \ldots, A_n$ are finite subsets of $\mathbb{Z}$ such that $|A_i| > k_i$ then there exist $a_1 \in A_1, \ldots, a_n \in A_n$ such that $f(a_1, \ldots, a_n) \neq 0$.

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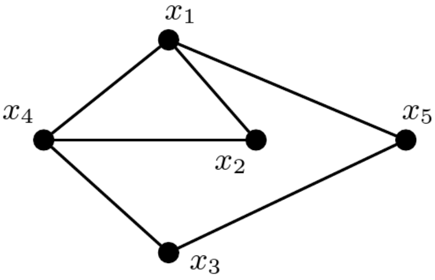

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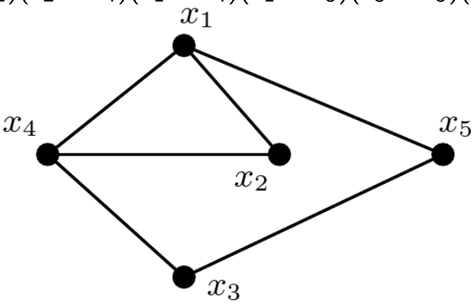

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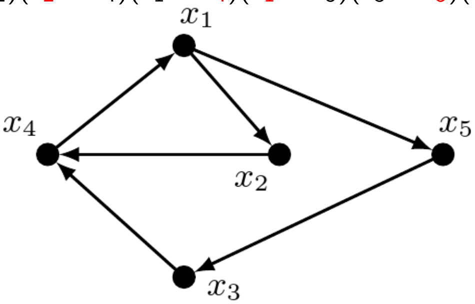

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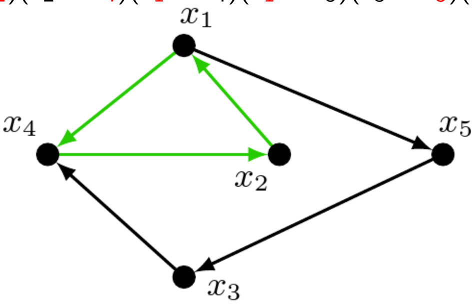

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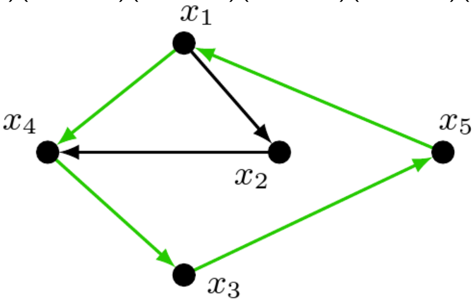

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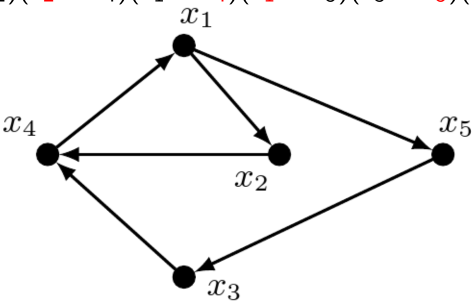

- an orientation D corresponds to a monomial
- coefficient is equal to $\pm|EE(D) - EO(D)|$
- outdegree in the graph corresponds to the degree in the monomial

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G is k-AT-critical if $AT(G) \geq k$ and $AT(H) < k$ for all proper induced subgraphs H of G.

AT-irreducibility

A graph G is *AT-reducible* to H if H is a nonempty induced proper subgraph of G which is f_H-AT where $f_H(v) = f(G) + d_H(v) - d_G(v)$ for all $v \in V(H)$.
If G is not AT-reducible to any nonempty induced subgraph, then it is *AT-irreducible*.

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$$L'(v) := L(v) - \pi(N(v) \cap V(G-H)) \text{ for } v \in H$$

$$|L'(v)| \geq |L(v)| - (d_G(v) - d_H(v)) = k - 1 + d_H(v) - d_G(v)$$

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- H is f_H-choosable so it is L'-colorable
- G is L-colorable – contradiction

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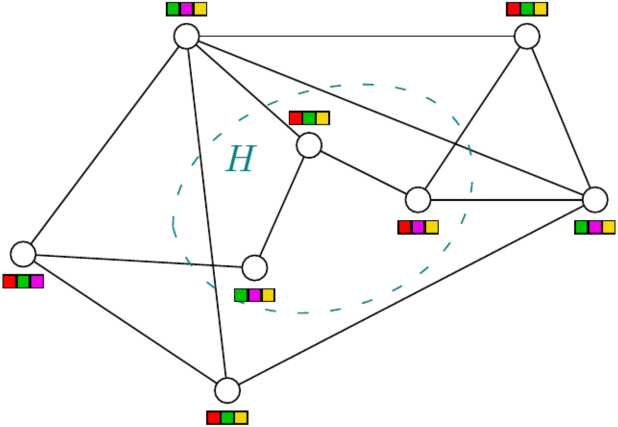

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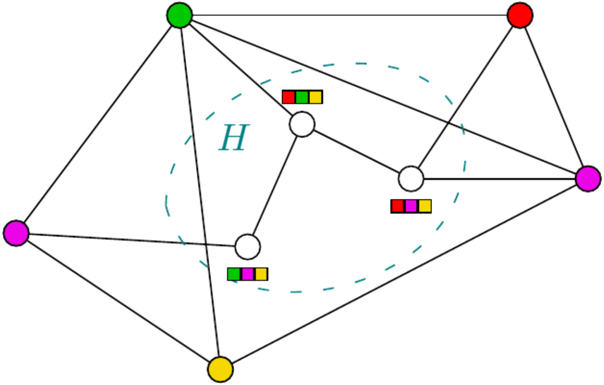

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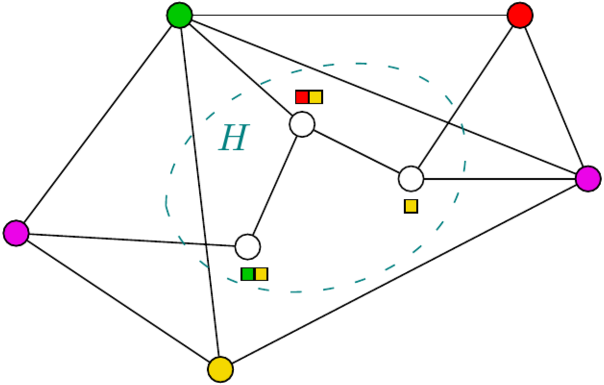

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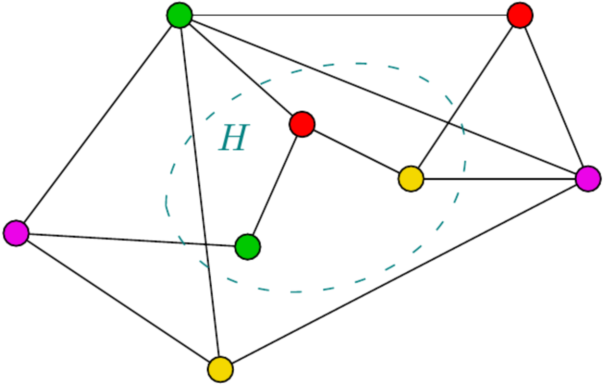

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Let G be a graph and $f : V(G) \to \mathbb{N}$. If H is an induced proper subgraph of G such that $G - H$ is $f\big|_{V(G-H)}$-AT and H is f_H-AT where $f_H(v) = f(v) + d_H(v) - d_G(v)$, then G is f-AT.

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- Take an orientation of $G - H$ demonstrating that it is $f\big|_{V(G-H)}$-AT
- Take an orientation of H demonstrating that it is f_H-AT
- Orient edges between $G - H$ and H into $G - H$
- For each $v \in V(H)$ the out-degree has increased by $d_G(v) - d_H(v)$

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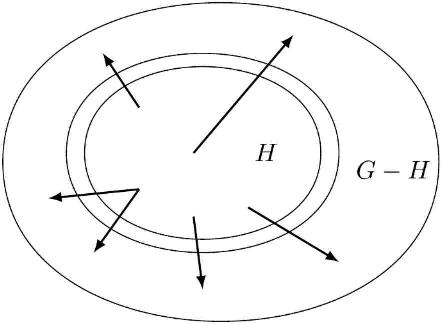

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Number of Eulerian subgraphs:

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$$(EE(H)EO(G - H) + EO(H)EE(G - H)) =$$
$$= (EE(H) - EO(H))(EE(G - H) - EO(G - H)) \neq 0$$

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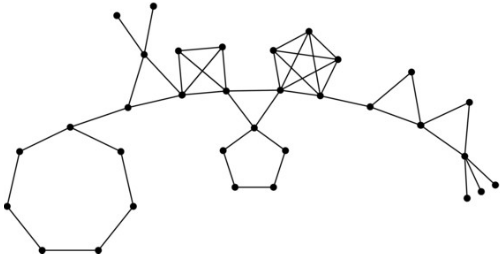
Gallai tree

A *Gallai tree* is a connected graph such that every block is either a clique or an odd cycle.



Source: `https://www.researchgate.net/figure/A-Gallai-tree-with-15-blocks_fig4_260483294`

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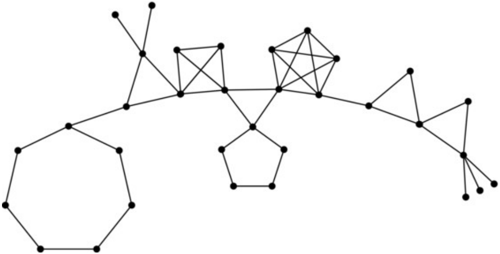

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Let $\mathcal{T}_k$ be the Gallai trees with maximum degree at most $k-1$, excepting K_k.

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Let $r \geq 0$, $k \geq r+4$ and $G \neq K_k$ be a graph with $x \in V(G)$ such that:

1. $G - x \in \mathcal{T}_k$; and
2. $d_G(x) \geq r+2$; and
3. $\left|N(x) \cap W^k(G-x)\right| \geq 1$; and
4. $d_G(v) \leq k-1$ for all $v \in V(G-x)$.

Then G is f-AT where $f(x) = d_G(x) - r$ and $f(v) = d_G(v)$ for all $v \in V(G-x)$.

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A 2-connected graph is either complete, an odd cycle or contains an induced even cycle with at most one chord.

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Connected graphs that are not d_0-choosable are Gallai trees.

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Lemma 3.4

For a connected graph G, the following are equivalent:

1. G is not a Gallai tree,
2. G contains an even cycle with at most one chord,
3. G is d_0-choosable,
4. G is d_0-AT,
5. G has an orientation D where $d_G(v) \geq d_D^+(v) + 1$ for all $v \in V(D)$, $EE(D) \in \{2, 3\}$ and $EO(D) \in \{0, 1\}$.

Lemma 3.4

2) G contains an even cycle with at most one chord $\implies$
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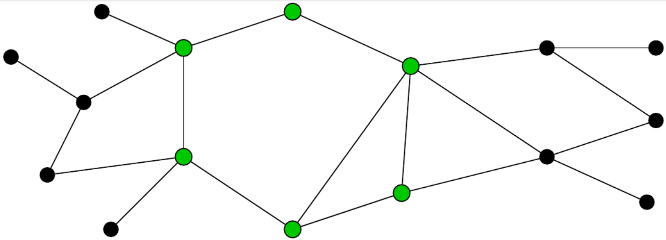

- Let H be an induced even cycle with at most one chord.
- Orient H clockwise and the chord arbitrarily.

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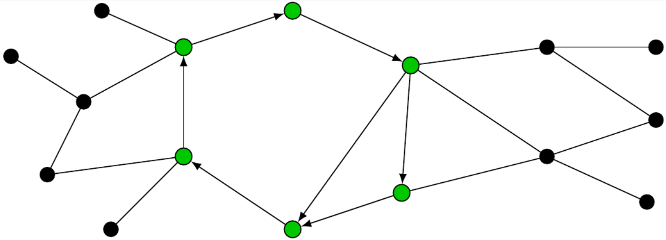
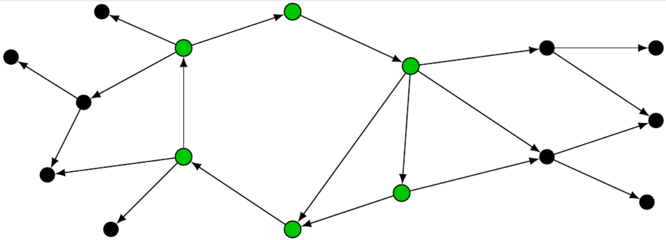


- Let H be an induced even cycle with at most one chord.
- Orient H clockwise and the chord arbitrarily.
- Contract H to x_H to obtain H'.
- Take a spanning tree of H' rooted at x_H and orient the edges away from x_H.
- Orientation is acyclic, except for H.

Lemma 3.4

2) G contains an even cycle with at most one chord $\implies$
5) G has an orientation D where $d_G(v) \geq d_D^+(v) + 1$ for all $v \in V(D)$, $EE(D) \in \{2,3\}$ and $EO(D) \in \{0,1\}$



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Lemma 3.7

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Let $k \geq 5$ and let G be a graph with $x \in V(G)$ such that:

1. $K_k \not\subseteq G$; and
2. $G - x$ has t components $H_1, H_2, \ldots, H_t$, and all are in $\mathcal{T}_k$; and
3. $d_G(v) \leq k - 1$ for all $v \in V(G - x)$; and
4. $\left|N(x) \cap W^k(H_i)\right| \geq 1$ for $i \in [t]$; and
5. $d_G(x) \geq t + 2$.

Then G is f-AT where $f(x) = d_G(x) - 1$ and $f(v) = d_G(v)$ for all $v \in V(G - x)$.

Lemma 3.9

For a graph G, let $\{X, Y\}$ be a partition of $V(G)$ and $k \geq 4$. Let $\mathcal{B}_k(X, Y)$ be the bipartite graph with one part Y and the other part the components of $G[X]$. Put an edge between $y \in Y$ and a component T of $G[X]$ iff $N(y) \cap W^k(T) \neq \emptyset$.

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Lemma 3.9

Let $k \geq 7$ and let G be a graph with $Y \subseteq V(G)$ such that:

1. $K_k \not\subseteq G$; and
2. the components of $G - Y$ are in $\mathcal{T}_k$; and
3. $d_G(v) \leq k - 1$ for all $v \in V(G - Y)$; and
4. with $\mathcal{B} := \mathcal{B}_k(V(G - Y), Y)$ we have $\delta(\mathcal{B}) \geq 3$.

Then G has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for all $v \in V(G' - Y)$.

Theorem 4.4

$$\alpha_k := \frac{1}{2} - \frac{1}{(k-1)(k-2)}$$

$$g_k(n, c) := \left(k - 1 + \frac{k-3}{(k-c)(k-1) + k - 3} \right) n$$

Theorem 4.4

If G is an AT-irreducible graph with $\delta(G) \geq 4$ and $\omega(G) \leq \delta(G)$, then $2\,\|G\| \geq g_{\delta(G)+1}(|G|\,, c)$ where $c := (\delta(G) - 2)\alpha_{\delta(G)+1}$ when $\delta(G) \geq 6$ and $c := (\delta(G) - 3)\alpha_{\delta(G)+1}$ when $\delta(G) \in \{4, 5\}$.

Corollaries

Corollary 5.1

For $k \geq 5$ and $G \neq K_k$ a k-list-critical graph, we have $2\,\|G\| \geq g_k(|G|\,, c)$ where $c := (k-3)\alpha_k$ when $k \geq 7$ and $c := (k-4)\alpha_k$ when $k \in \{5, 6\}$.

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Corollary 5.2

For $k \geq 5$ and $G \neq K_k$ an online k-list-critical graph, we have $2\,\|G\| \geq g_k(|G|\,, c)$ where $c := (k-3)\alpha_k$ when $k \geq 7$ and $c := (k-4)\alpha_k$ when $k \in \{5, 6\}$.

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For $k \geq 5$ and $G \neq K_k$ an online k-list-critical graph, we have $2\,\|G\| \geq g_k(|G|\,, c)$ where $c := (k-3)\alpha_k$ when $k \geq 7$ and $c := (k-4)\alpha_k$ when $k \in \{5, 6\}$.

Corollary 5.3

For $k \geq 5$ and $G \neq K_k$ a k-AT-critical graph, we have $2\,\|G\| \geq g_k(|G|\,, c)$ where $c := (k-3)\alpha_k$ when $k \geq 7$ and $c := (k-4)\alpha_k$ when $k \in \{5, 6\}$.