A note on degree-constrained subgraphs

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f-factor

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Definition

f-factor is a spanning subgraph H of G in which $d_H(v) \in f(v)$ for all $v \in V$.

Theorem 1 (Shirazi and Verstraëte)

Theorem

Let $G = (V, E)$ be a graph and suppose that f satisfies

$$|f(v)| > \lceil d(v)/2 \rceil$$

for every $v \in V$.
Then G has an f-factor.

Combinatorial Nullstellensatz

Theorem

Let $g \in \mathbb{F}[X_1, X_2, ..., X_n]$ be a polynomial, and suppose the coefficient of the monomial $\prod_{i=1}^{n} X_i^{t_i}$ in g is non-zero, where $t_1 + t_2 + ... + t_n$ is the total degree of g.
Then, for any sets $S_1, S_2, ..., S_n \subset \mathbb{F}$ with $|S_1| > t_1, |S_2| > t_2, ..., |S_n| > t_n$, there exists $x \in S_1 \times S_2 \times ... \times S_n$ such that $g(x) \neq 0$.

F-avoiding graphs

Definition

If $F(v) \subseteq \mathbb{N}$ is a set of forbidden degrees for every $v \in V$, then a subgraph $G' = (V, E')$ of G is called **F-avoiding** if $d_{G'}(v) \notin F(v)$ for all $v \in V$.

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Proof (induction on the number of edges):
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Base case:
If 0 is not a forbidden degree at any node, then the empty subgraph $(V, \emptyset)$ is F-avoiding.

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Then $\varrho_D(t) \geq |F(t)| \geq 1$

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There is an edge $e = st$ of G for which $\overrightarrow{e}$ is directed toward t.

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Since $|F^-(t)| = |F(t)| - 1$, $\varrho_{D^-}(v) \geq |F^-(v)|$ holds for every node v.
By induction, there is an F-avoiding subgraph G'' of G^-.
By the construction of F^-, the subgraph $G' := G'' + e$ of G is F-avoiding.

$\square$

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Every undirected graph G has an orientation D in which $\varrho_D(v) \geq \lfloor d_G(v)/2 \rfloor$ for every node v.

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Every undirected graph G has an orientation D in which $\varrho_D(v) \geq \lfloor d_G(v)/2 \rfloor$ for every node v.
Therefore **Theorem 2** implies **Theorem 1**.

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Theorem 3

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Let $G = (V, E)$ be an undirected graph, and let F satisfy

$$\sum_{v \in V} |F(v)| < |E|$$

and $0 \notin F(v)$.
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By induction, there is a nonempty F^--avoiding subgraph G' of G^-.
As $d_{G'}(t) < d_G(t)$, this G' is also F-avoiding.

Bibliography

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2. Shirazi, H.; Verstraëte, J. *A note on polynomials and f-factors of graphs*, 2008.